PHY 256 Lecture 6

Composite Systems
+
Quantum entanglement

The composite system axiom

$\mathcal{H}_A \otimes \mathcal{H}_B$

Tensor product

$\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = \dim \mathcal{H}_A \cdot \dim \mathcal{H}_B$

$\dim(\underset{m}{\mathbb{C}^m} \otimes \underset{n}{\mathbb{C}^n}) = mn$

$|\psi_A\rangle \in \mathcal{H}_A \quad |\psi_B\rangle \in \mathcal{H}_B$

$|\psi_A\rangle \otimes |\psi_B\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

$|A_i\rangle \quad i \in \{1, \dots, m\} \quad \mathcal{H}_A = \mathbb{C}^m$

$|B_j\rangle \quad j \in \{1, \dots, n\} \quad \mathcal{H}_B = \mathbb{C}^n$

$|A_i\rangle \otimes |B_j\rangle, \quad i \in \{1, \dots m\}, \; j \in \{1, \dots n\}$

$\lambda \in \mathbb{C} \quad |\psi_A\rangle \in \mathcal{H}_A \quad |\psi_B\rangle \in \mathcal{H}_B$

$\lambda(|\psi_A\rangle \otimes |\psi_B\rangle) = (\lambda|\psi_A\rangle) \otimes |\psi_B\rangle$
$= |\psi_A\rangle \otimes (\lambda|\psi_B\rangle)$

$|\psi_A\rangle, |\phi_A\rangle \in \mathcal{H}_A \quad |\theta_B\rangle \in \mathcal{H}_B$

$(|\psi_A\rangle + |\phi_A\rangle) \otimes |\theta_B\rangle =$
$|\psi_A\rangle \otimes |\theta_B\rangle + |\phi_A\rangle \otimes |\theta_B\rangle$

$\mathbb{C}^2 \otimes \mathbb{C}^3 \quad \underset{\mathbb{C}^2}{|\psi_A\rangle} \otimes \underset{\mathbb{C}^3}{|\psi_B\rangle} \neq \cancel{|\psi_B\rangle \otimes |\psi_A\rangle}$

$O_A \in \mathcal{H}_A \quad O_B \in \mathcal{H}_B$

$O_A \otimes O_B \quad \mathcal{H}_A \otimes \mathcal{H}_B$

$(\underset{2\times 2}{O_A} \otimes \underset{3\times 3}{O_B})(\underset{2}{|\psi_A\rangle} \otimes \underset{3}{|\psi_B\rangle}) = (O_A|\psi_A\rangle) \otimes (O_B|\psi_B\rangle)$

$\mathbb{C}^2 \otimes \mathbb{C}^3$

$(\underset{2}{\langle\psi_A|} \otimes \underset{3}{\langle\psi_B|})(\underset{2}{|\phi_A\rangle} \otimes \underset{3}{|\phi_B\rangle}) = \langle\psi_A|\phi_A\rangle\langle\psi_B|\phi_B\rangle$

$O_A, P_A \in \mathcal{H}_A \quad O_B, P_B \in \mathcal{H}_B$

$(O_A \otimes O_B)(P_A \otimes P_B) = O_A P_A \otimes O_B P_B$

$\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \cdots$

$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2 \quad |\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbb{C}^2$

$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \psi_1 |\phi\rangle \\ \psi_2 |\phi\rangle \end{pmatrix} = \begin{pmatrix} \psi_1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\ \psi_2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \psi_1\phi_1 \\ \psi_1\phi_2 \\ \psi_2\phi_1 \\ \psi_2\phi_2 \end{pmatrix} \in \mathbb{C}^4$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ 2 \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}$

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}$

$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & & \vdots \\ A_{21}B_{11} & A_{21}\cdots & & \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 \cdot \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} & 1 \cdot \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \\ 2 \cdot \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} & 0 \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 6 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{4\times 4}$

Quantum Entanglement

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|0\rangle \otimes |0\rangle \quad |0\rangle \otimes |1\rangle$
$|1\rangle \otimes |0\rangle \quad |1\rangle \otimes |1\rangle$

$|00\rangle \equiv |0\rangle \otimes |0\rangle$
$|01\rangle \equiv |0\rangle \otimes |1\rangle$

$S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$|0\rangle \quad +\frac{1}{2}$
$|1\rangle \quad -\frac{1}{2}$

$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$

Separable State

$|\psi_A\rangle \otimes |\psi_B\rangle$

$|\psi\rangle = |0\rangle \otimes |0\rangle \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to 0$

$|\psi\rangle_A = |0\rangle \quad |\psi_B\rangle = |0\rangle$

$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \qquad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to 0$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Entangled State

$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to -\frac{1}{2}$

$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$

$|\psi\rangle = (\beta_0|0\rangle + \beta_1|1\rangle) \otimes (\gamma_0|0\rangle + \gamma_1|1\rangle)$

$\alpha_{00} = \beta_0\gamma_0 \quad \alpha_{01} = \beta_0\gamma_1$
$\alpha_{10} = \beta_1\gamma_0 \quad \alpha_{11} = \beta_1\gamma_1$

$\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} = \beta_0\gamma_0\beta_1\gamma_1 - \beta_0\gamma_1\beta_1\gamma_0 = 0$

$\alpha = \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}$

$\det\alpha = \alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}$

separable $\longleftrightarrow \det\alpha = 0$

The Bell States (EPR)

$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\boxed{|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)}$

$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |0\rangle \; |1\rangle \quad +\frac{1}{2} \; -\frac{1}{2} \quad Z$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \qquad |+\rangle \; |-\rangle \quad +\frac{1}{2} \; -\frac{1}{2} \quad X$

$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$