PHYS 1P22/92

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Spring 2024

10. Rotational Motion and Angular Momentum

10.1 Angular Acceleration

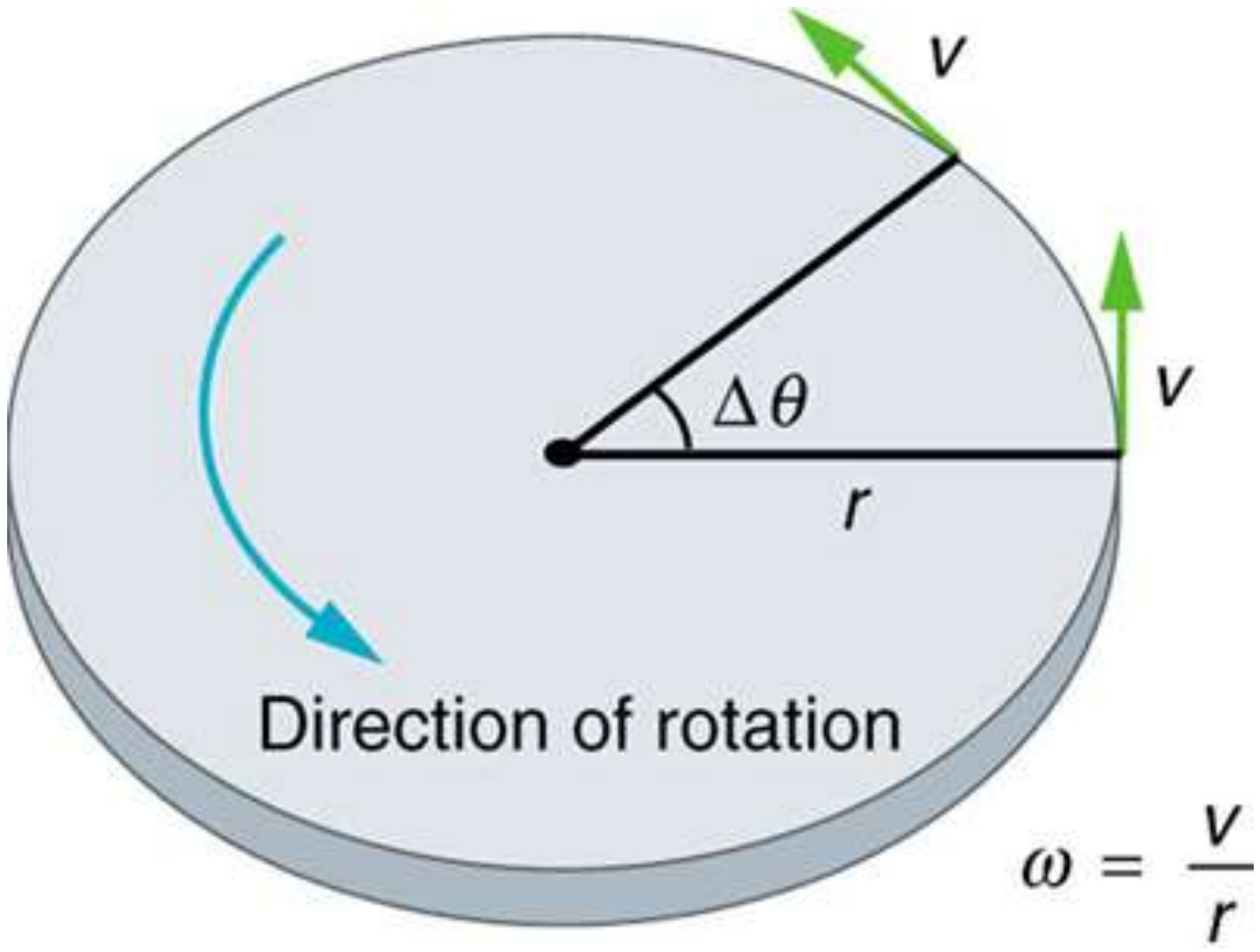
Reminder: Uniform circular motion

- Textbook chapter 6 (should review)
- Angle as function of time: $\theta(t)$
- Angular velocity:

$$\omega \equiv \frac{\Delta\theta}{\Delta t} \qquad \text{(really a derivative, } \omega \equiv \frac{\mathrm{d}\theta}{\mathrm{d}t}\text{)}$$

- Relationship to linear velocity v and radius r:

$$v = r\omega$$


v
v
Δθ
r
Direction of rotation
ω = v/r

Angular acceleration

- Linear acceleration a: rate of change of linear velocity

$$a \equiv \frac{\Delta v}{\Delta t} \qquad \text{(really a derivative, } a \equiv \frac{\mathrm{d}v}{\mathrm{d}t}\text{)}$$

Units: velocity per second $= (\mathrm{m/s})/\mathrm{s} = \mathrm{m/s^2}$

- Angular acceleration α (alpha): rate of change of angular velocity

$$\alpha \equiv \frac{\Delta \omega}{\Delta t} \qquad \text{(really a derivative, } \alpha \equiv \frac{\mathrm{d}\omega}{\mathrm{d}t}\text{)}$$

Units: $\mathrm{rad/s^2}$

How to solve a problem

- Always calculate the **complete analytical expression** first, and only plug in numbers in the end!
- Unit conversions cannot be part of the analytical expression, since they are part of the number.
- Separate the numerical calculation into pure numbers and pure units.
- Final answer should have the same number of significant figures as the **least precise** numerical quantity in the question.

Problem: A bicycle wheel is spinning from rest to 250 rpm in 5.00 s. Calculate the angular acceleration in rad/s^2.

Analytical solution:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{\Delta t}$$

Numerical solution: $\omega_1 = 0, \omega_2 = 250 \text{ rpm}, \Delta t = 5 \text{ s}$

$$\alpha = \frac{\omega_2 - \omega_1}{\Delta t} \approx \frac{(250 - 0) \text{ rpm}}{5 \text{ s}} = \frac{250 \text{ rpm}}{5 \text{ s}}$$

Convert units:

$$1 \text{ revolution} = 2\pi \text{ radians}, \qquad 1 \text{ minute} = 60 \text{ seconds}$$

$$\Longrightarrow 1 \text{ rpm} = \text{revolutions per minute} = (2\pi \text{ rad})/60\text{s}$$

$$\alpha = \frac{250 \cdot (2\pi \text{ rad}/60 \text{ s})}{5 \text{ s}} = \frac{250 \cdot 2\pi}{60 \cdot 5} \cdot \frac{\text{rad}}{\text{s}^2} \approx 5.24 \text{ rad/s}^2$$

Problem: If we slam on the brakes, causing an angular acceleration of $-87.3\ \mathrm{rad/s^2}$, how long does it take the wheel to stop?

Analytical solution:

$$\alpha = \frac{\Delta\omega}{\Delta t} \quad \Longrightarrow \quad \Delta t = \frac{\Delta\omega}{\alpha} = \frac{\omega_2 - \omega_1}{\alpha}$$

Numerical solution: $\omega_1 = 250\ \mathrm{rpm}, \omega_2 = 0, \alpha = -87.3\,\frac{\mathrm{rad}}{\mathrm{s^2}}$

$$\Delta t = \frac{\omega_2 - \omega_1}{\alpha} \approx \frac{(0 - 250)\ \mathrm{rpm}}{-87.3\ \mathrm{rad/s^2}} = \frac{-250 \cdot (2\pi\ \mathrm{rad}/60\ \mathrm{s})}{-87.3\ \mathrm{rad/s^2}}$$

$$= \frac{-250 \cdot 2\pi}{-87.3 \cdot 60} \cdot \frac{\mathrm{rad} \cdot \mathrm{s^2}}{\mathrm{rad} \cdot \mathrm{s}} = 0.300\ \mathrm{s}$$

Linear vs. angular acceleration

$$v = r\omega$$

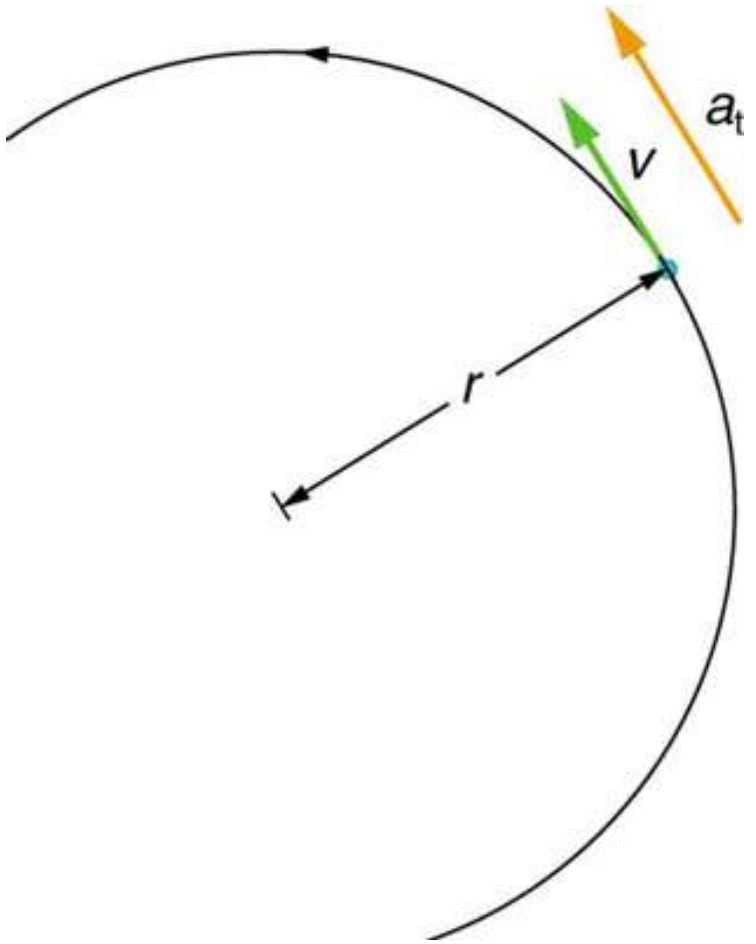
$$a \equiv \frac{\Delta v}{\Delta t}, \qquad \alpha \equiv \frac{\Delta \omega}{\Delta t}$$

r is constant, so $\Delta r = 0$:

$$\Delta v = \Delta(r\omega) = r\Delta\omega$$

$$a = \frac{\Delta v}{\Delta t} = r\frac{\Delta \omega}{\Delta t} = r\alpha$$

$$\Longrightarrow a = r\alpha$$



Problem: A powerful motorcycle can accelerate from 0 to 30.0 m/s (about 108 km/h) in 4.20 s. What is the angular acceleration of its 0.320-m-radius wheels?

Analytical solution:

$$a = r\alpha \quad \Longrightarrow \quad \alpha = \frac{a}{r}$$

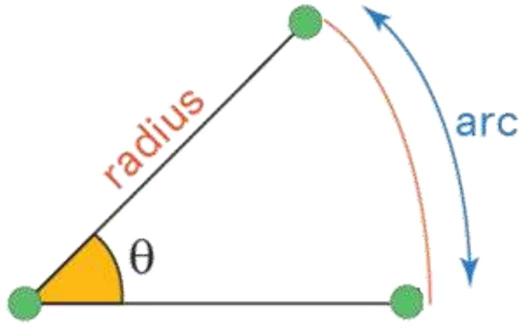
$$a = \frac{\Delta v}{\Delta t} \quad \Longrightarrow \quad \alpha = \frac{\Delta v/\Delta t}{r} = \frac{\Delta v}{r\Delta t} = \frac{v_2 - v_1}{r\Delta t}$$

Numerical solution:

$$\alpha = \frac{v_2 - v_1}{r\Delta t} = \frac{(30 - 0)\ \mathrm{m/s}}{0.32\ \mathrm{m} \cdot 4.2\ \mathrm{s}} = 22.3\,\frac{\mathrm{rad}}{\mathrm{s}^2}$$

Radians are dimensionless (rad $\equiv$ 1), no need to convert units!

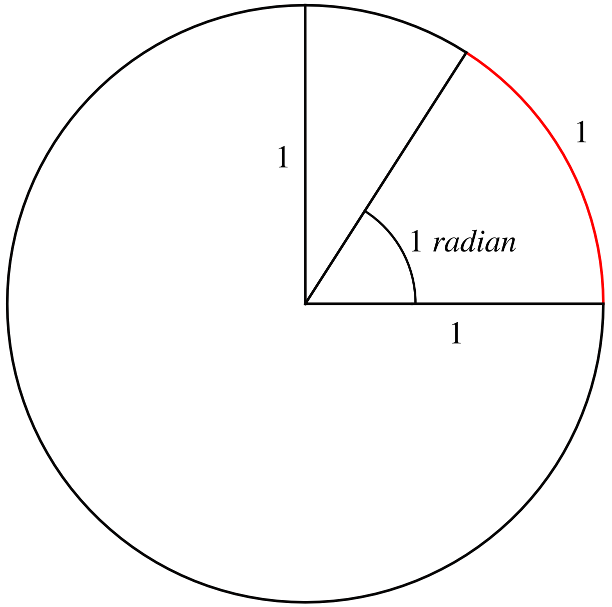
Quick review of radians



- If x is the arc length and r is the radius, then the angle in radians is:

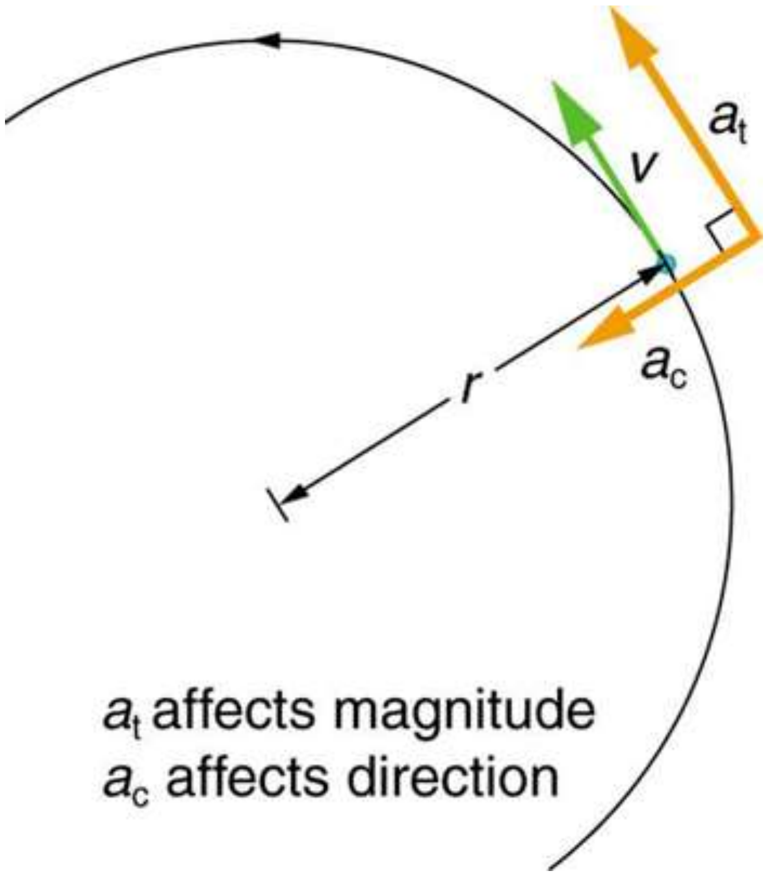
$$\theta \equiv \frac{x}{r}$$

- Note: x and r both measured in meters, so in terms of units, rad $=$ m/m $= 1$!
- 1 rad: if $x = r$.
- Full circle: $x = 2\pi r$ (circumference), so $\theta = 2\pi$.
- Angles are specified in radians by default. Degrees must be denoted explicitly with °.



Tangential vs. centripetal acceleration

- Tangential acceleration $a_t = r\alpha$: change in speed.
- Centripetal acceleration $a_c = v^2/r$: change in direction.
- Always perpendicular to each other.



Analogous quantities: linear vs. angular

Linear/Translational	Angular/Rotational	Relationship
Position x	Angle θ	$x = r\theta$
Velocity v	Angular velocity ω	$v = r\omega$
Acceleration a	Angular acceleration α	$a = r\alpha$

10.2 Kinematics of Rotational Motion

Velocity and acceleration

- Constant a, starting velocity v_0:

$$v = v_0 + at$$

- Since $v = r\omega$ and $a = r\alpha$:

$$r\omega = r\omega_0 + r\alpha t$$

- Cancel r:

$$\omega = \omega_0 + \alpha t$$

Note: Constant α.

Adding position / angle

- With position (x_0 = initial position, a still constant):

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

(Derived using calculus, by integrating $v = v_0 + at$. See chapter 2 for non-calculus derivation.)

- Angular version, using $x = r\theta$:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

Equations without time

- Sometimes it's convenient to eliminate t from the equation:

$$v^2 = v_0^2 + 2a(x - x_0)$$

(See chapter 2 for derivation)

- Angular version:

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Analogous equations: linear vs. angular

(Constant a and α)

Linear/Translational	Angular/Rotational
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

10.3 Dynamics of Rotational Motion: Rotational Inertia

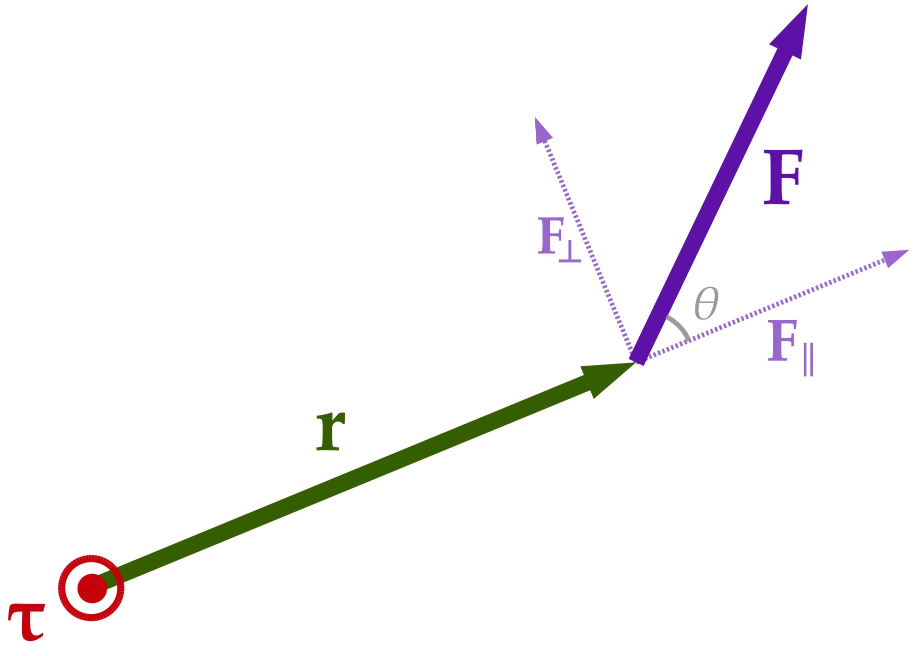
Torque

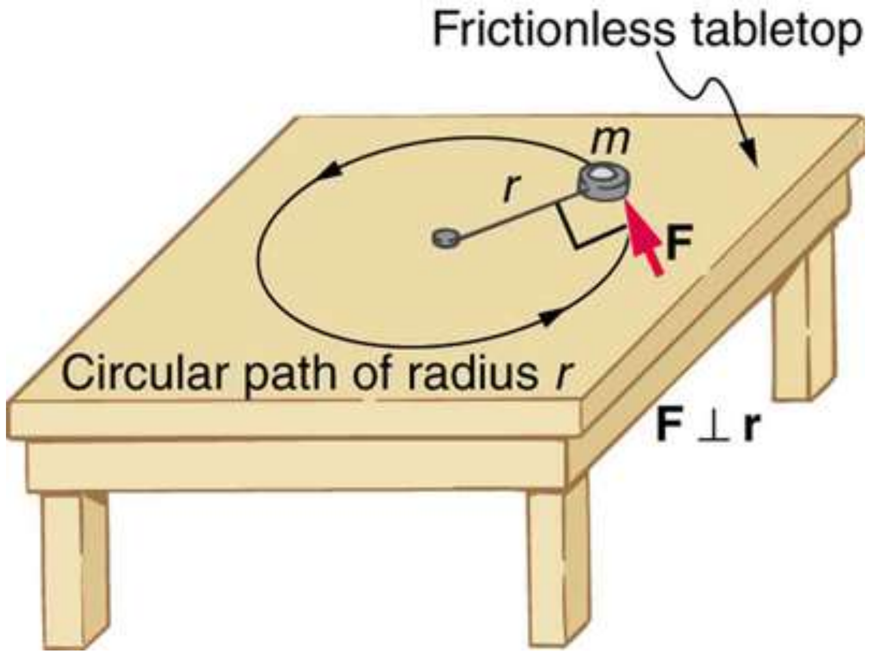
- Recall from chapter 9 that torque τ is the angular analogue of force. In vector terms:

$$\boldsymbol{\tau} \equiv \boldsymbol{r} \times \boldsymbol{F}, \qquad \tau = |\boldsymbol{\tau}| = rF\sin\theta$$

- In circular motion, the radius and force are perpendicular, so:

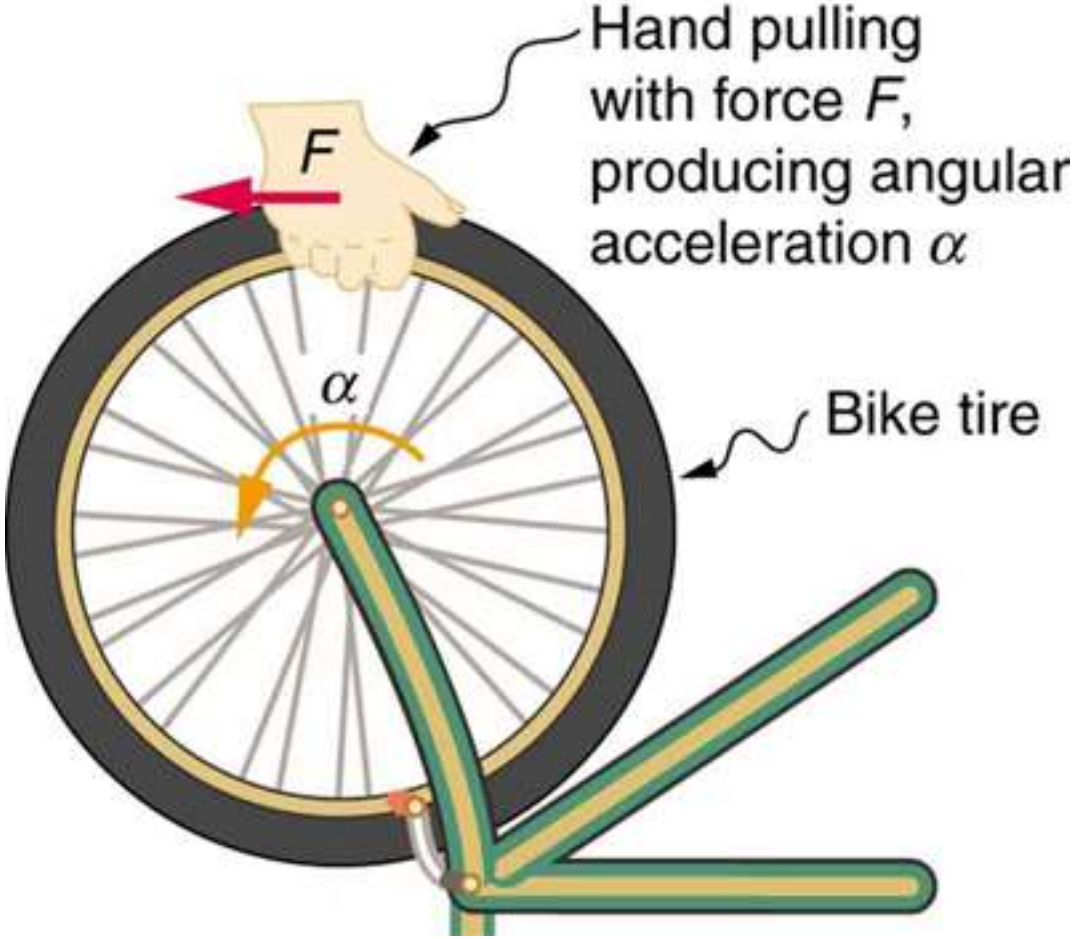
$$\theta = \frac{\pi}{2} \implies \sin\theta = 1 \implies \tau = rF$$





Spinning a wheel

- Intuitively:
  - More force = more acceleration
  - More massive wheel = less acceleration
  - Smaller radius (push closer to center) = less acceleration



Torque

- Newton's 2nd law:

$$F = ma$$

- Since $a = r\alpha$:

$$F = mr\alpha$$

- Torque $\tau = rF$. Multiply both sides by r:

$$Fr = mr^2\alpha \quad \Longrightarrow \quad \tau = mr^2\alpha$$

- Define moment of inertia $I \equiv mr^2$:

$$\tau = I\alpha$$

Analogous quantities

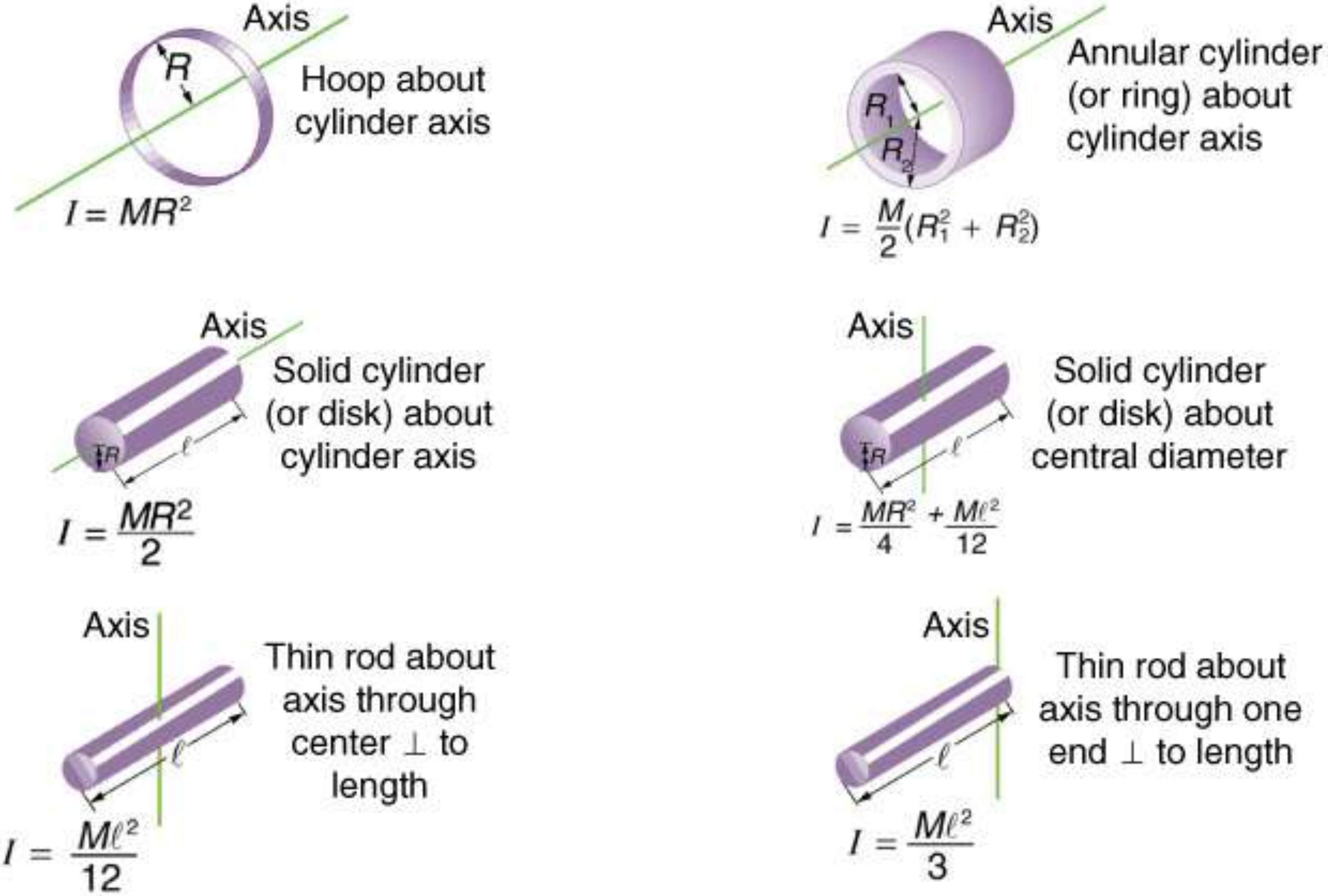
Linear/Translational	Angular/Rotational
2nd law: $F = ma$	2nd law: $\tau = I\alpha$
Force F	Torque τ
Acceleration a	Angular acceleration α
Mass m	Moment of inertia $I \equiv mr^2$

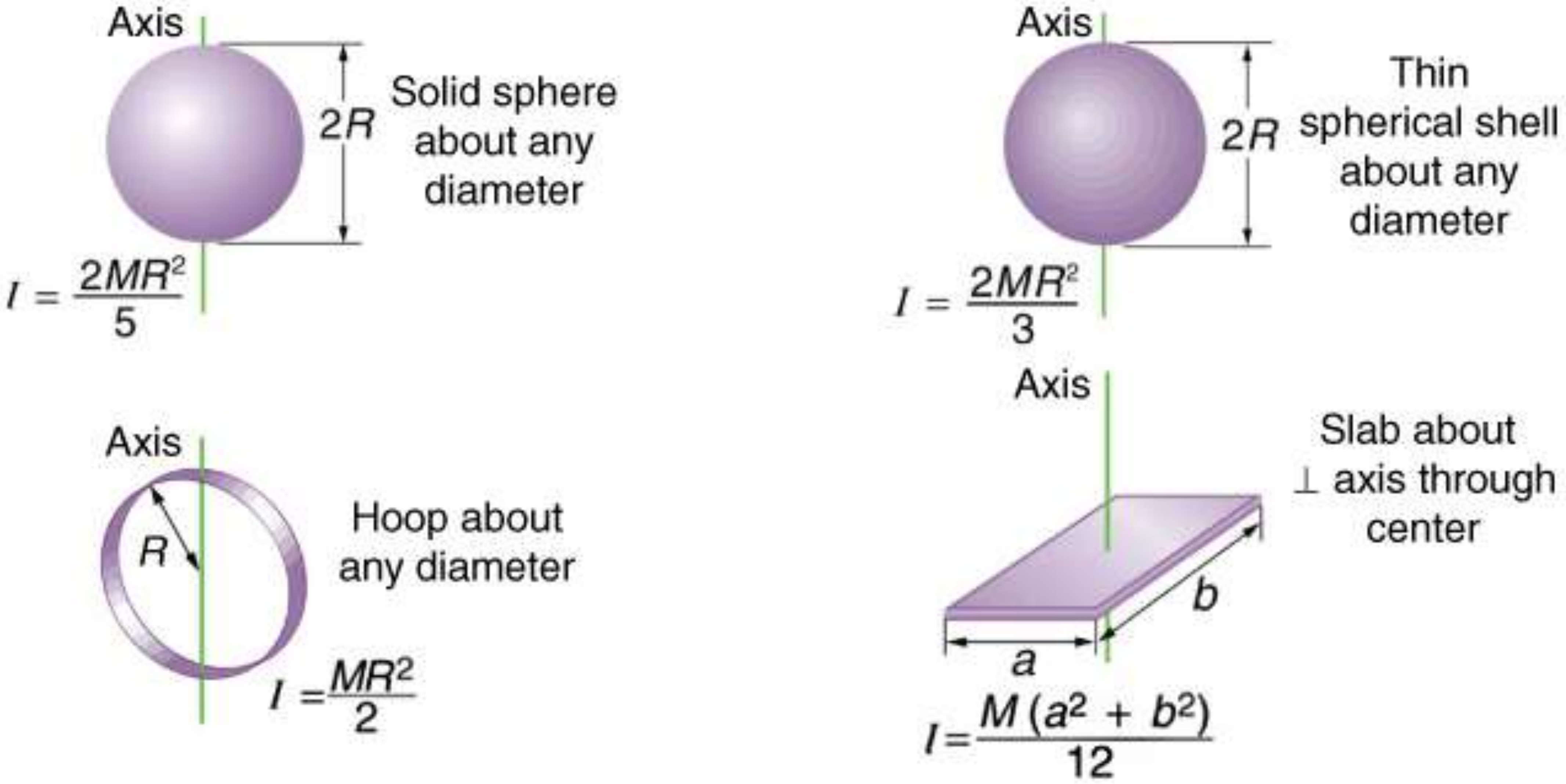
Moment of inertia for non-point mass

- For a point mass, $I = mr^2$.
- For other objects, we sum over all the point masses, each with its own m and r^2.
- **Without calculus:** $I = \sum_i m_i r_i^2$, where i is an index enumerating all the masses. For example, for two masses $i = 1,2$:

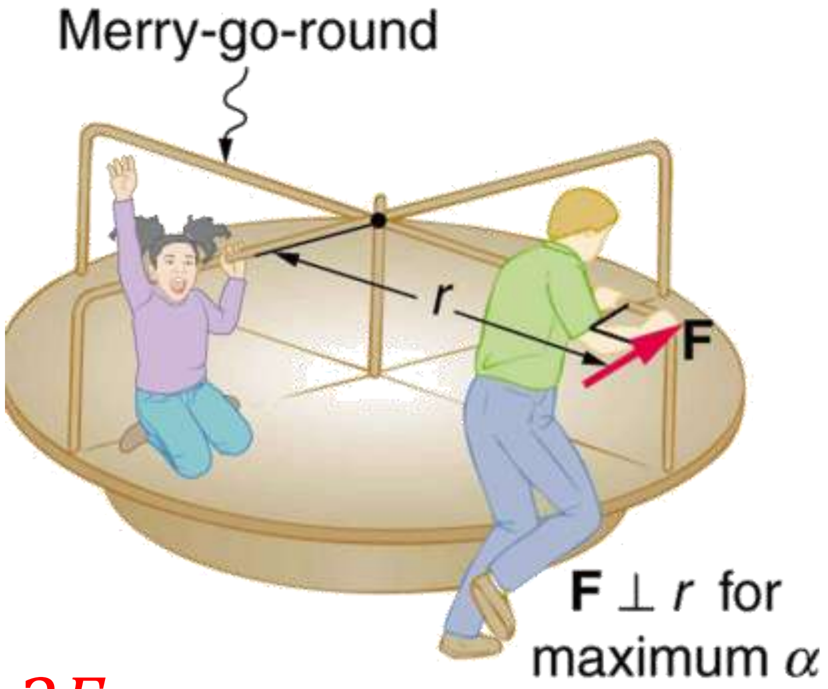
$$I = m_1 r_1^2 + m_2 r_2^2$$

  - **With calculus:** Using integrals (for continuous objects)


Axis
R
Hoop about cylinder axis
I = MR²
Axis
R₁
R₂
Annular cylinder (or ring) about cylinder axis
I = M/2 (R₁² + R₂²)
Axis
R
ℓ
Solid cylinder (or disk) about cylinder axis
I = MR²/2
Axis
R
ℓ
Solid cylinder (or disk) about central diameter
I = MR²/4 + Mℓ²/12
Axis
ℓ
Thin rod about axis through center ⊥ to length
I = Mℓ²/12
Axis
ℓ
Thin rod about axis through one end ⊥ to length
I = Mℓ²/3


Axis
2R
Solid sphere about any diameter
I = 2MR²/5
Axis
2R
Thin spherical shell about any diameter
I = 2MR²/3
Axis
R
Hoop about any diameter
I = MR²/2
Axis
Slab about ⊥ axis through center
b
a
I = M (a² + b²)/12

Problem: A father pushing a playground merry-go-round exerts a force of 250 N at the edge of the 50.0-kg merry-go-round, which has a 1.50 m radius. Calculate the angular acceleration produced when no one is on the merry-go-round.



Analytical solution:

- Torque: $\tau = rF$
- Moment of inertia (from table): $I = \frac{1}{2}mr^2$
- Newton's 2nd law:

$$\tau = I\alpha \quad \Longrightarrow \quad \alpha = \frac{\tau}{I} = \frac{rF}{\frac{1}{2}mr^2} = \frac{2F}{mr}$$

Problem: A father pushing a playground merry-go-round exerts a force of 250 N at the edge of the 50.0-kg merry-go-round, which has a 1.50 m radius. Calculate the angular acceleration produced when no one is on the merry-go-round.

Numerical solution:

$$\alpha = \frac{2F}{mr} \approx \frac{2 \cdot 250\ \mathrm{N}}{50\ \mathrm{kg} \cdot 1.5\ \mathrm{m}} = \frac{2 \cdot 250}{50 \cdot 1.5} \frac{\mathrm{N}}{\mathrm{kg} \cdot \mathrm{m}} \approx 6.67 \frac{\mathrm{N}}{\mathrm{kg} \cdot \mathrm{m}}$$

$$\text{Units: } \mathrm{N} = \frac{\mathrm{kg} \cdot \mathrm{m}}{\mathrm{s}^2} \quad \Longrightarrow \quad \frac{\mathrm{N}}{\mathrm{kg} \cdot \mathrm{m}} = \frac{1}{\mathrm{kg} \cdot \mathrm{m}} \frac{\mathrm{kg} \cdot \mathrm{m}}{\mathrm{s}^2} = \frac{1}{\mathrm{s}^2} = \frac{\mathrm{rad}}{\mathrm{s}^2}$$

$$\alpha \approx 6.67\ \mathrm{rad/s}^2$$

Problem: Calculate the angular acceleration produced when an 18.0-kg child sits 1.25 m away from the center.

Analytical solution: Add the two moments of inertial together (child = point mass):

$$I_{\text{disk}} = \frac{1}{2} m_{\text{disk}} r_{\text{disk}}^2, \qquad I_{\text{child}} = m_{\text{child}} r_{\text{child}}^2$$

Therefore (r for the torque $= r_{\text{disk}}$!):

$$\alpha = \frac{\tau}{I} = \frac{r_{\text{disk}} F}{\frac{1}{2} m_{\text{disk}} r_{\text{disk}}^2 + m_{\text{child}} r_{\text{child}}^2}$$

Problem: Calculate the angular acceleration produced when an 18.0-kg child sits 1.25 m away from the center.

Numerical solution:

$$\alpha = \frac{r_{\text{disk}} F}{\frac{1}{2} m_{\text{disk}} r_{\text{disk}}^2 + m_{\text{child}} r_{\text{child}}^2}$$

$$= \frac{1.5\ \text{m} \cdot 250\ \text{N}}{\frac{1}{2}(50\ \text{kg})(1.5\ \text{m})^2 + (18\ \text{kg})(1.25\ \text{m})^2}$$

$$= 4.44\ \frac{\text{m} \cdot \text{N}}{\text{kg} \cdot \text{m}^2}$$

$$= 4.44\ \text{rad/s}^2$$

10.4 Rotational Kinetic Energy: Work and Energy Revisited

Reminder: work

- Definition of work W (chapter 7):

$$W \equiv \boldsymbol{F} \cdot \boldsymbol{s} = Fs\cos\theta$$

$\boldsymbol{F}$ = force vector being applied ($F = |\boldsymbol{F}|$)

$\boldsymbol{s}$ = displacement vector ($s = |\boldsymbol{s}|$)

$\cdot$ = dot product of vectors

- Example: Applying a force of $F = 1\ \mathrm{N}$ along a distance of $s = 1\ \mathrm{m}$, with the force parallel to the distance ($\theta = 0$, $\cos\theta = 1$), results in work of $W = 1\ \mathrm{N} \cdot \mathrm{m}$.

Rotational work

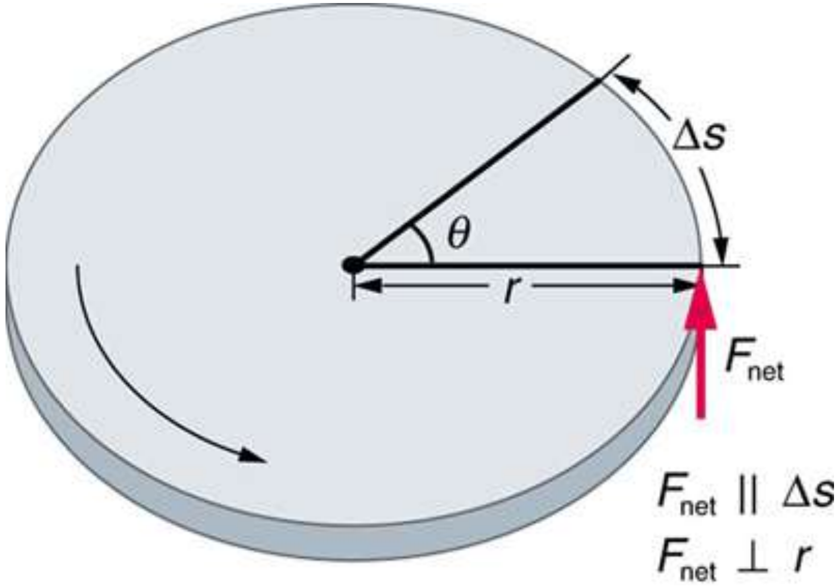
- Applying a force on the disk, parallel to the tangent, along an arc length s:

$$W = Fs$$

- Torque is $\tau = rF$, so $F = \tau/r$. Arc length is $s = r\theta$:

$$W = \frac{\tau}{r} \cdot r\theta = \tau\theta$$

Therefore rotational work is $W = \tau\theta$.



Reminder: kinetic energy

- Equation of motion without time: $v^2 - v_0^2 = 2a(x - x_0)$
  - Displacement: $\mathrm{s} = x - x_0$
  - Rename v_0 to v_1 (initial velocity) and v to v_2 (final velocity)

$$v_2^2 - v_1^2 = 2as \quad \Longrightarrow \quad s = \frac{v_2^2 - v_1^2}{2a}$$

- Plug into definition of work along with Newton's 2nd law $F = ma$:

$$W = Fs = (ma)\left(\frac{v_2^2 - v_1^2}{2a}\right) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

This is the work-energy theorem: work is the change in kinetic energy.

Note about notation…

- The textbook uses KE for the kinetic energy and PE for potential energy.
- This is a very confusing notation since it looks like K times E or P times E.
- In the lectures we will use the (standard) notation E_k for kinetic energy and E_p for potential energy.

Rotational kinetic energy

- From the work-energy theorem, linear kinetic energy for a particle moving at velocity v is $E_k = \frac{1}{2}mv^2$.
- As usual, there are angular analogues, which can be derived similarly:

$$W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2, \qquad E_k = \frac{1}{2}I\omega^2$$

- Again, I is analogous to mass m and ω to linear velocity v.

Total kinetic energy

- Sometimes there is both linear and angular kinetic energy, for example for a rolling object.
- The total kinetic energy (linear + angular) is:

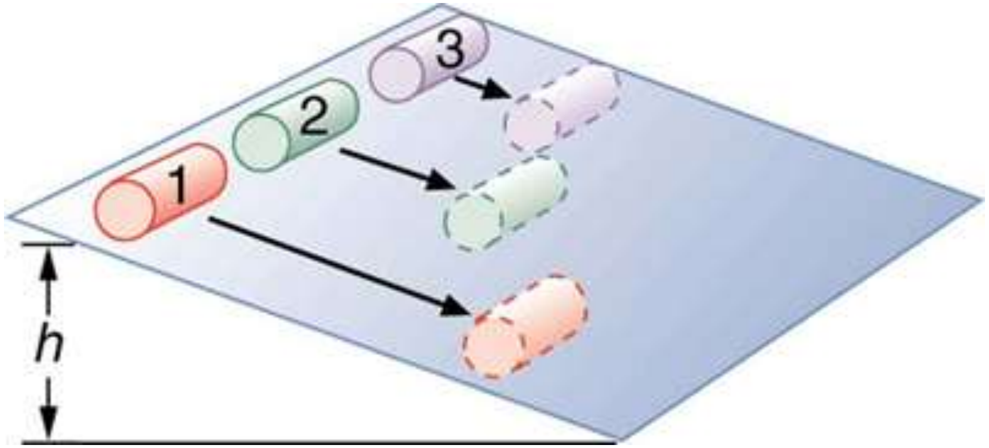
$$E_k = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Problem: Calculate the final speed of a solid cylinder that rolls down a 2.00-m-high incline. The cylinder starts from rest, has a mass of 0.750 kg, and has a radius of 4.00 cm.

Analytical solution:

- The cylinder starts at rest, with only potential energy $E_p = mgh$.
- It ends at $h = 0$, so with no potential energy.
- The potential energy was converted to linear + angular kinetic energy:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$



Analytical solution (cont.):

- The moment of inertia for a cylinder is $I = \frac{1}{2}mr^2$.
- We want to isolate the speed v. The angular velocity is $\omega = v/r$.

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2$$

Divide by m:

$$gh = \frac{3}{4}v^2 \quad \Longrightarrow \quad v = \left(\frac{4}{3}gh\right)^{1/2}$$

Numerical solution:

$$v = \left(\frac{4}{3} g h\right)^{1/2}$$

$$\approx \left(\frac{4}{3}\left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)(2\ \mathrm{m})\right)^{1/2}$$

$$= \left(\left(\frac{4}{3} \cdot 9.8 \cdot 2\right)\left(\frac{\mathrm{m}}{\mathrm{s}^2} \cdot \mathrm{m}\right)\right)^{1/2}$$

$$\approx \left(26.1\,\frac{\mathrm{m}^2}{\mathrm{s}^2}\right)^{1/2} \approx 5.11\ \mathrm{m/s}$$

Analogous quantities

Linear/Translational	Angular/Rotational
Work $W = Fs$	Work $W = \tau\theta$
Kinetic energy $E_k = \frac{1}{2}mv^2$	Kinetic energy $E_k = \frac{1}{2}I\omega^2$

10.5 Angular Momentum and Its Conservation

Angular momentum

- Linear momentum p is defined as

$$p \equiv mv$$

- Quiz: How can we define angular momentum (denoted L)?
- Answer: Since I is analogous to m and ω is analogous to v…

$$L = I\omega$$

Example: Angular momentum of Earth

- Earth is a sphere, so (from the table):

$$I = \frac{2}{5}mr^2 \quad \Longrightarrow \quad L = I\omega = \frac{2}{5}mr^2\omega$$

  - $m \approx 5.98 \times 10^{24}\ \mathrm{kg}$
  - $r \approx 6.38 \times 10^{6}\ \mathrm{m}$
  - $\omega \approx 1$ revolution per day $\approx 2\pi\ \mathrm{rad}/(24 \times 60 \times 60\ \mathrm{s}) \approx 7.27 \times 10^{-5}\ \mathrm{rad/s}$

$$L \approx \frac{2}{5}(5.98 \times 10^{24}\ \mathrm{kg})(6.38 \times 10^{6}\ \mathrm{m})^2 \left(7.27 \times 10^{-5}\ \frac{\mathrm{rad}}{\mathrm{s}}\right)$$

$$\approx 7.08 \times 10^{33}\ \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}$$

Newton's 2nd law

- In linear motion, force is the change in momentum over time:

$$F = \frac{\Delta p}{\Delta t}$$

  - Since $p = mv$, **if the mass is constant** we get the simpler form

$$F = m\frac{\Delta v}{\Delta t} = ma$$

- Quiz: What will be the analogous law for angular motion?
- Answer:

$\tau = \frac{\Delta L}{\Delta t}$, and if I is constant: $\tau = I\frac{\Delta \omega}{\Delta t} = I\alpha$ as we found before!

Conservation of angular momentum

- We learned in chapter 8 that linear momentum is conserved.
  - To prove this, note that if $F = 0$ then

$$F = \frac{\Delta p}{\Delta t} \quad \Longrightarrow \quad \Delta p = 0$$

So momentum never changes (it is conserved) unless a force is applied.

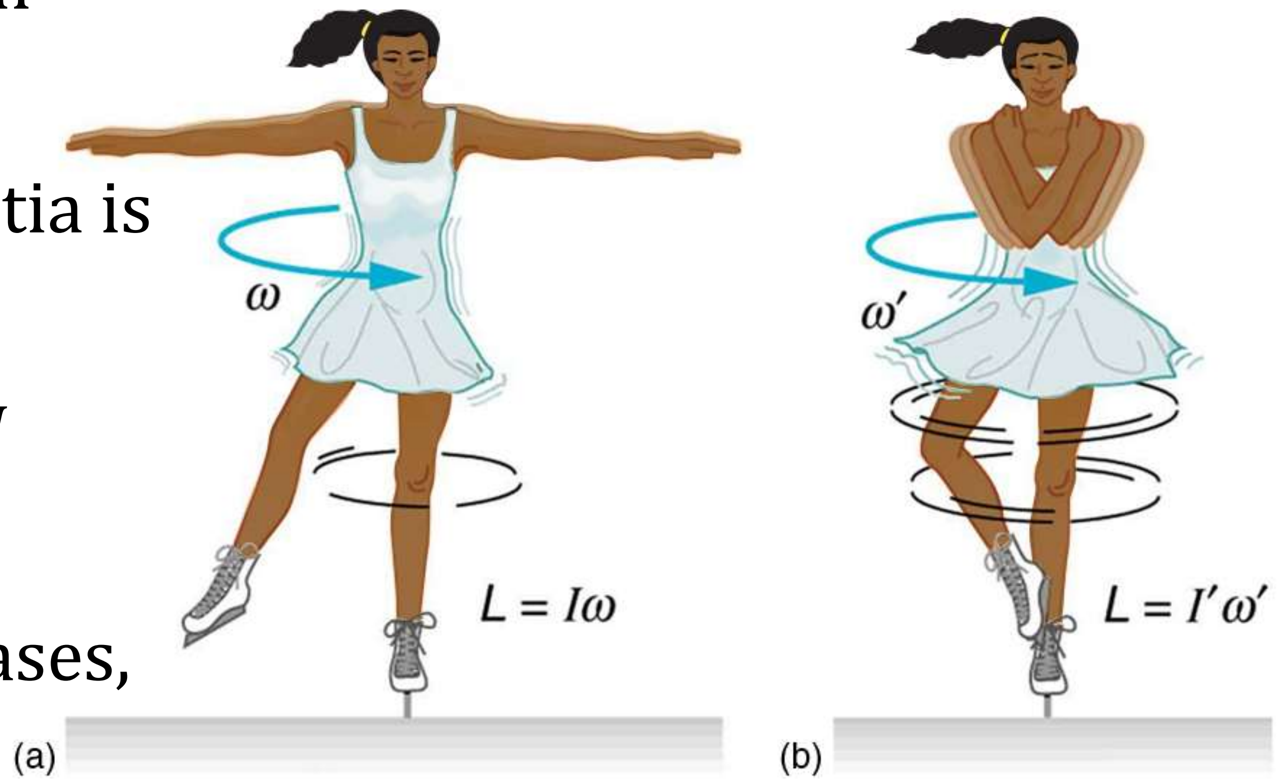
- Similarly, angular momentum is conserved if $\tau = 0$:

$$\tau = \frac{\Delta L}{\Delta t} \quad \Longrightarrow \quad \Delta L = 0$$

Conservation of angular momentum

- This is why the Earth keeps spinning around itself and around the Sun!
- As long as no external torque is applied to it, its speed of rotation will never change.
  - There are actually some minor torques being applied, e.g. the gravity of the Moon, slowing down the Earth's rotation $\approx$ 65.7 ns/day!

- Another famous example is ice skating.
- The skater can keep spinning for a long time, because there is almost no friction.
- Also, by pulling her arms in, she can increase her angular speed.
- This is because the moment of inertia is proportional to r^2.
- By decreasing r (pulling arms in), I also decreases.
- Since $L = I\omega$ is constant, if I decreases, ω must increase.



Video

- Here's a video demonstrating the use of angular momentum in ice skating:

https://youtu.be/FmnkQ2ytlO8