PHYS 1P22/92

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12. Fluid Dynamics

12.1 Flow Rate and Its Relation to Velocity

- The (volumetric) flow rate Q is the volume of fluid passing through a surface per unit time:

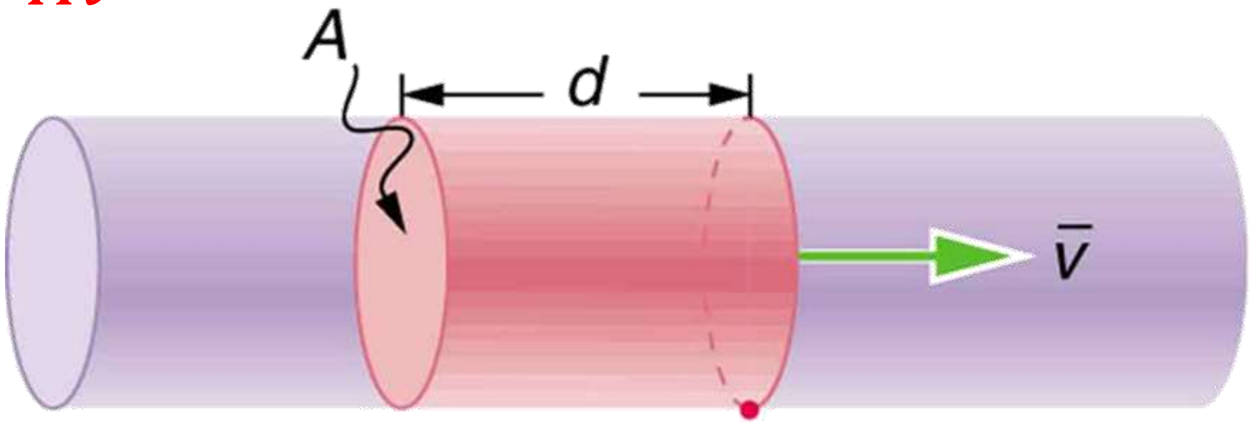
$$Q \equiv \frac{\Delta V}{\Delta t}, \qquad \text{or just } Q \equiv \frac{V}{t}$$

- If A is the cross-sectional area and a length d of fluid passes through in time t then

$$V = Ad \quad \Longrightarrow \quad Q = \frac{V}{t} = \frac{Ad}{t}$$

- Let $\bar{v} \equiv d/t$ be the average fluid velocity, then:

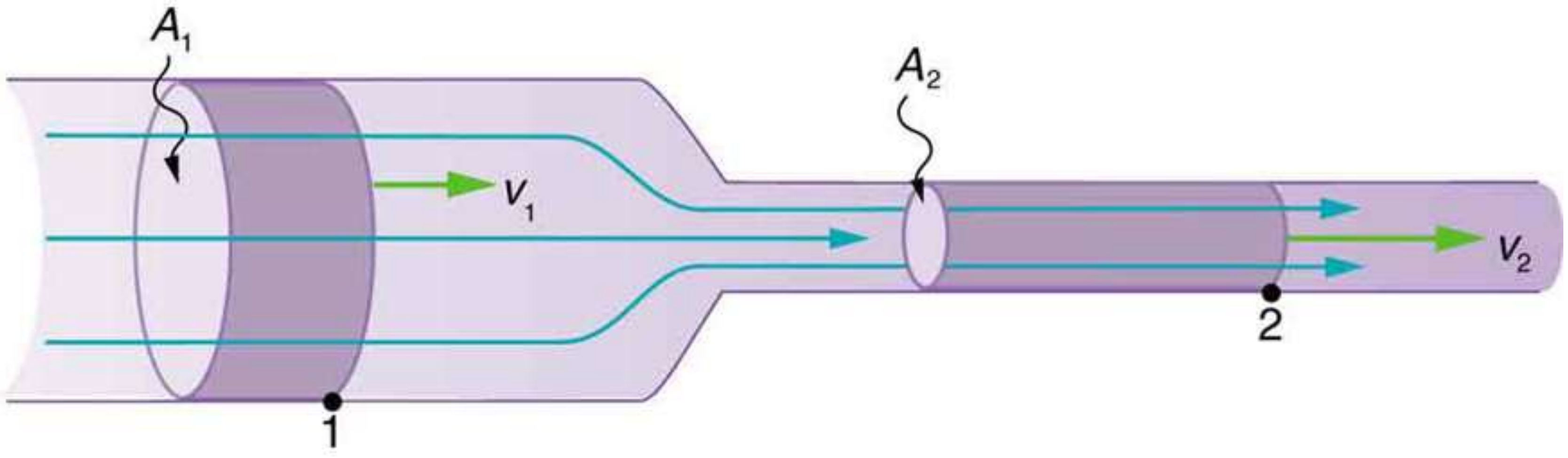
$$Q = A\bar{v}$$



- Consider an **incompressible** fluid flowing through a pipe with flow rate Q.
- Pop Quiz: Given $Q = A\bar{v}$, what happens when the cross-sectional area decreases?
- Answer: Since the flow rate Q must be constant:

$$Q_1 = Q_2 \quad \Longrightarrow \quad A_1\bar{v}_1 = A_2\bar{v}_2$$

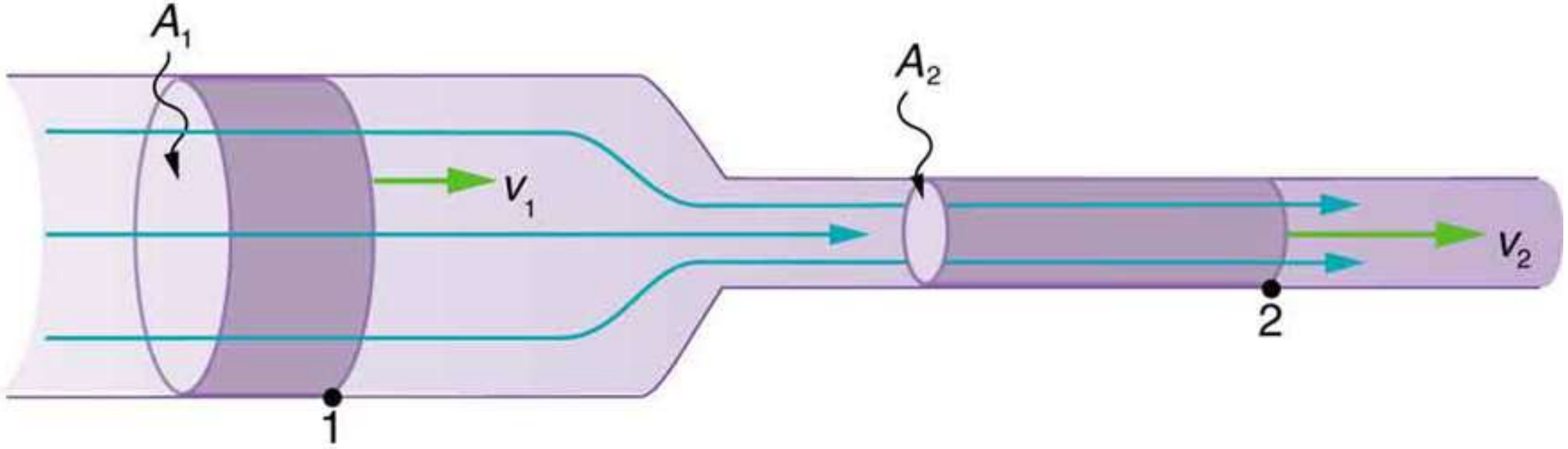
- This is called the continuity equation. If A decreases then $\bar{v}$ must increase, and vice versa.
- Example: Flow through a nozzle.



- Problem: A nozzle with radius r_2 is attached to a garden hose with radius $r_1 > r_2$. The flow rate is Q. Calculate the speed of the water in the hose and nozzle.
- Solution: We have $A_1 = \pi r_1^2$ and $A_2 = \pi r_2^2$. Then

$$Q = A_1 \bar{v}_1 \quad \Longrightarrow \quad \bar{v}_1 = \frac{Q}{A_1} = \frac{Q}{\pi r_1^2}$$

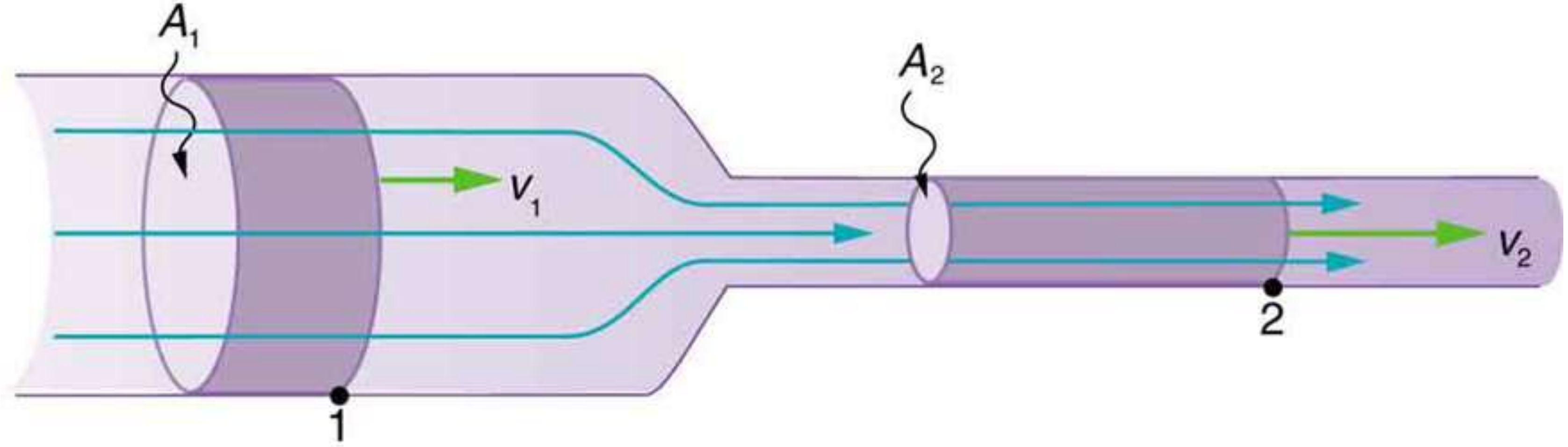
$$A_1 \bar{v}_1 = A_2 \bar{v}_2 \quad \Longrightarrow \quad \bar{v}_2 = \frac{A_1}{A_2} \bar{v}_1 = \frac{\pi r_1^2}{\pi r_2^2} \left(\frac{Q}{\pi r_1^2} \right) = \frac{Q}{\pi r_2^2}$$



- Problem: Calculate the speeds if $r_2 \approx 0.250$ cm, $r_1 \approx 0.900$ cm, and $Q \approx 0.500$ L/s.
- Solution: $1\text{ L} = 10^{-3}\text{ m}^3$, $1\text{ cm} = 10^{-2}\text{ m}$, so:

$$\bar{v}_1 = \frac{Q}{\pi r_1^2} \approx \frac{0.5\text{ L/s}}{\pi(0.9\text{ cm})^2} = \frac{0.5 \times 10^{-3}\text{ m}^3/\text{s}}{\pi(0.9 \times 10^{-2}\text{ m})^2} = \frac{0.5 \times 10^{-3}}{\pi \cdot (0.9 \times 10^{-2})^2}\frac{\text{m}^3/\text{s}}{\text{m}^2}$$
$$\approx 1.96\text{ m/s}$$

$$\bar{v}_2 = \frac{Q}{\pi r_2^2} \approx \cdots = \frac{0.5 \times 10^{-3}}{\pi \cdot (0.25 \times 10^{-2})^2}\frac{\text{m}^3/\text{s}}{\text{m}^2} \approx 25.5\text{ m/s}$$



Speed vs. radius

- This is a general result: speed is inversely proportional to the square of the radius.

$$v \propto \frac{1}{r^2}$$

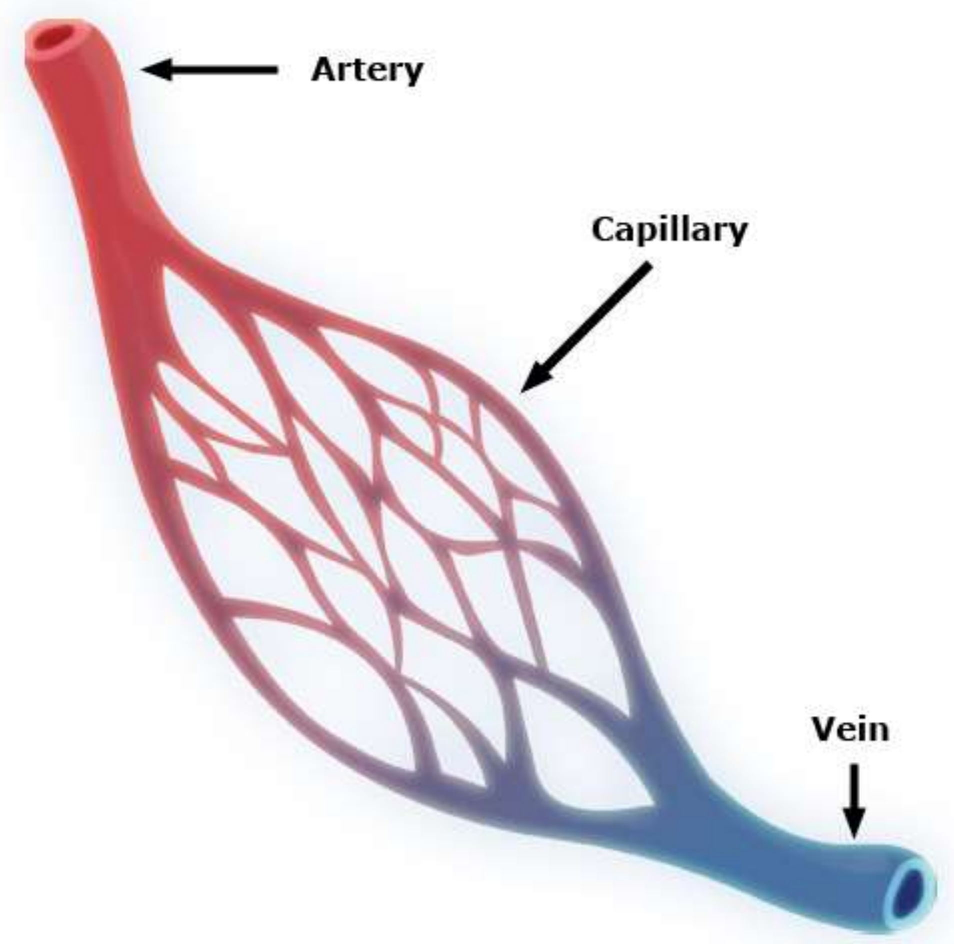
- Example: Blowing a candle with open mouth vs. pursed lips.

Branching flow

- Sometimes the flow branches, e.g.: cardiovascular system.
- In this case, the continuity equation $A_1\bar{v}_1 = A_2\bar{v}_2$ applies for the sum of the flow rates:

$$n_1 A_1 \bar{v}_1 = n_2 A_2 \bar{v}_2$$

- n_1 and n_2 are the number of branches on sides 1 and 2.



Problem: Blood flows through the large aorta and smaller capillaries. When the aorta flow rate is $Q_1 \approx 5.0$ L/min, the speed of blood in the capillaries is $\bar{v}_2 \approx 0.33$ mm/s. If the average diameter of a capillary is $D_2 \approx 8.0$ μm, calculate the number of capillaries in the blood circulatory system.

Solution: $n_1 = 1$ (only one aorta). From $n_1 A_1 \bar{v}_1 = n_2 A_2 \bar{v}_2$ and $Q_1 = A_1 \bar{v}_1$:

$$n_2 = \frac{n_1 (A_1 \bar{v}_1)}{A_2 \bar{v}_2} = \frac{n_1 Q_1}{A_2 \bar{v}_2} \approx \frac{(1) \cdot (5.0\ \mathrm{L/min})}{\pi \cdot (0.5 \cdot 8.0\ \mu\mathrm{m})^2 \cdot 0.33\ \mathrm{mm/s}}$$

$$= \frac{(5 \times 10^{-3}\ \mathrm{m}^3/60\ \mathrm{s})}{\pi \cdot (0.5 \cdot 8.0 \times 10^{-6}\ \mathrm{m})^2 \cdot 0.33 \times 10^{-3}\ \mathrm{m/s}}$$

$$= \frac{5 \times 10^{-3}/60}{\pi \cdot (0.5 \cdot 8.0 \times 10^{-6})^2 \cdot 0.33 \times 10^{-3}} \frac{\mathrm{m}^3/\mathrm{s}}{\mathrm{m}^2 \cdot \mathrm{m/s}} \approx 5.0 \times 10^9$$

12.2 Bernoulli’s Equation

Fluid and energy

- We saw that if A decreases, $\bar{v}$ increases. Where does the extra kinetic energy come from?
- Energy is conserved:

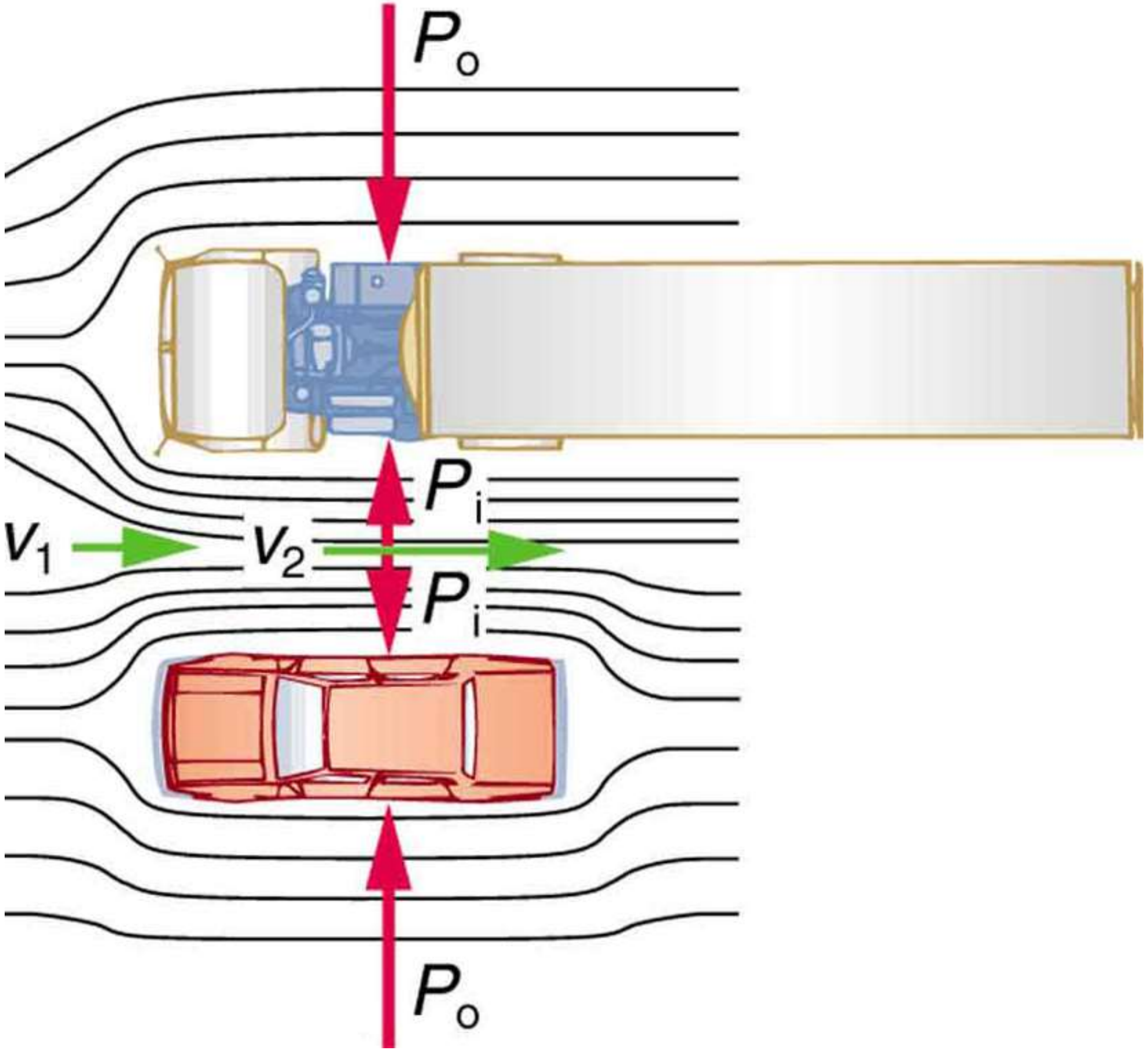
$$E_{\text{kinetic}} + E_{\text{potential}} = \text{constant}$$

- If kinetic energy increases, potential energy must decrease.
- Pressure contributes to potential energy through work:

$$W = Fx = PAx$$

- So pressure must decrease if velocity increases.

- Examples:
  - Shower curtain moves into shower. High-velocity stream inside shower creates lower pressure.
  - Car driving next to a truck is pulled toward it.



Bernoulli's equation

- An incompressible, frictionless fluid with pressure P, density ρ, and velocity v, at height h, satisfies:

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

- This is actually just conservation of energy per unit volume:

$$\rho \equiv \frac{m}{V} \quad \Longrightarrow \quad PV + \frac{1}{2}mv^2 + mgh = V\left(P + \frac{1}{2}\rho v^2 + \rho g h\right)$$

- $\frac{1}{2}mv^2$ is kinetic energy, mgh is gravitational potential energy, and

$$\frac{V}{A} = x \quad \Longrightarrow \quad PV = \frac{F}{A}V = Fx = W$$

Bernoulli's equation

- If we take a "snapshot" of the LHS at two points 1 and 2, we have:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

- Can be used to find quantities at 2 based on quantities at 1.
- Can rewrite in terms of **differences**:

$$(P_2 - P_1) + \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) = 0$$

- Sometimes one of the 3 terms vanishes. Let's see some examples.

Static fluid

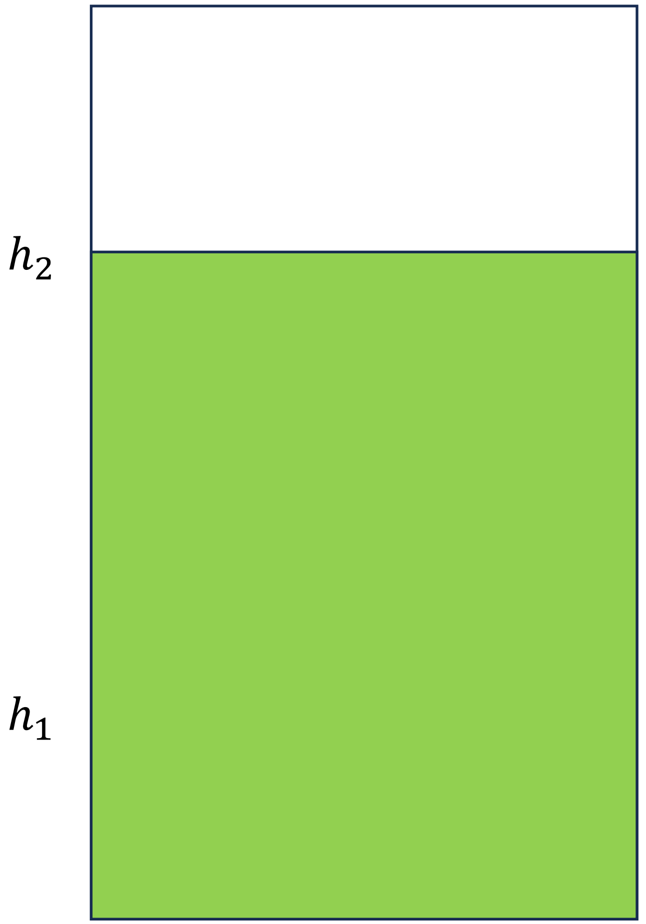
- Start from $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$
- For a static fluid $v = 0$, so:

$$P + \rho g h = \text{constant}$$

or $$P_1 + \rho g h_1 = P_2 + \rho g h_2$$

or $$P_1 - P_2 = \rho g (h_2 - h_1)$$

- If $P_2 = 0$ and $h_2 - h_1 = h$ we get the formula $P = \rho g h$ from ch. 11.



Constant depth

- Start from

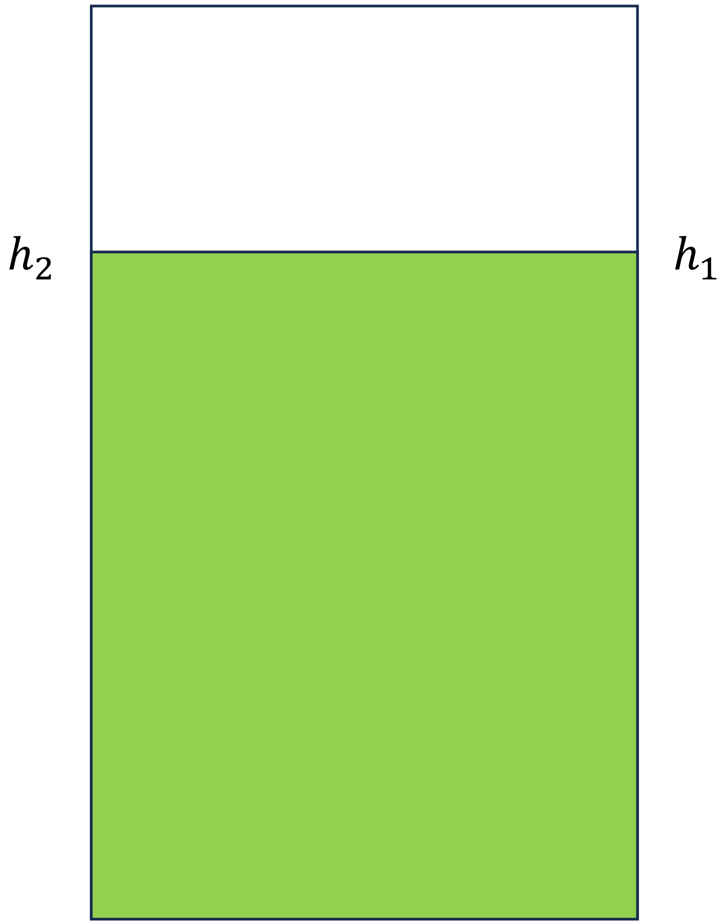
$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

- At constant depth $h_1 = h_2$, so terms cancel:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

or $$P + \frac{1}{2}\rho v^2 = \text{constant}$$

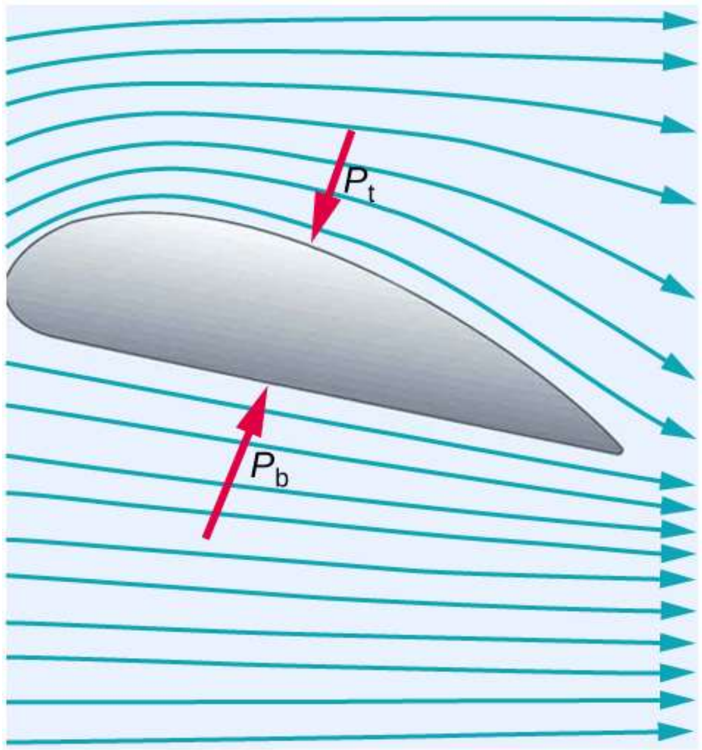
- We see that if v increases, P must decrease, or vice versa.

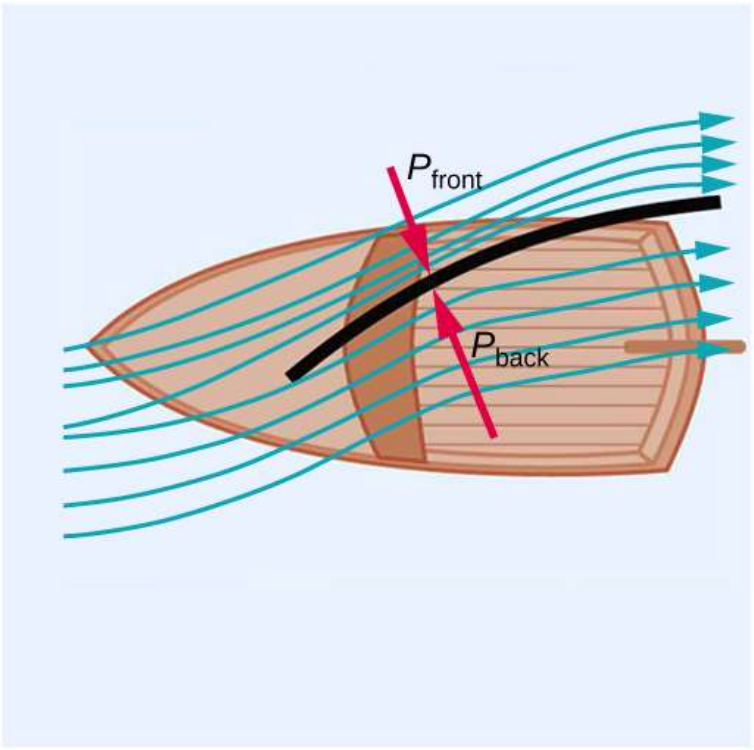


- Air flows faster over the upper surface of a plane’s wing, causing lower pressure and a net upward force or lift.

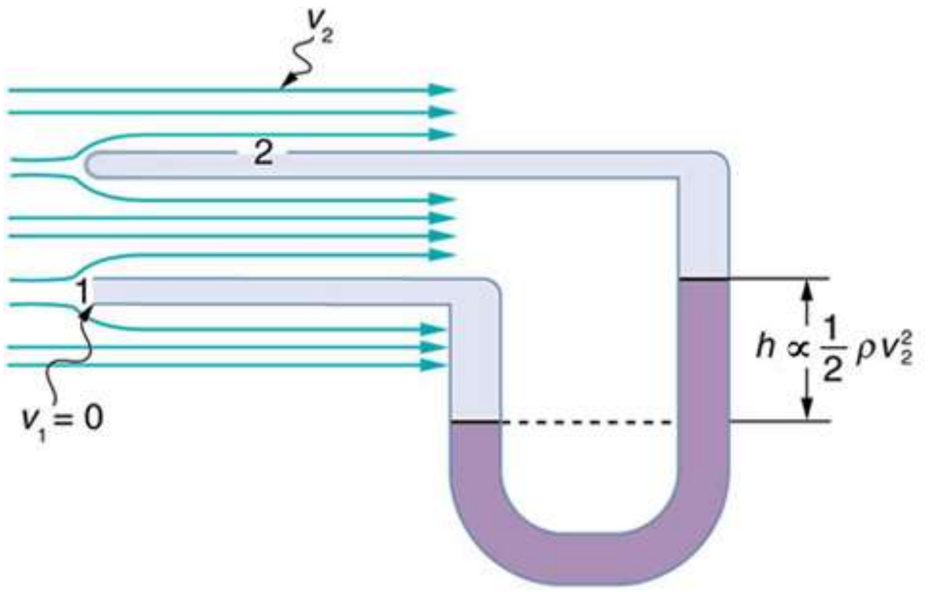
Warning! There is a common misconception that the speed difference is because the upper surface is longer. That is incorrect. The real reason is complicated and we won’t cover it here.

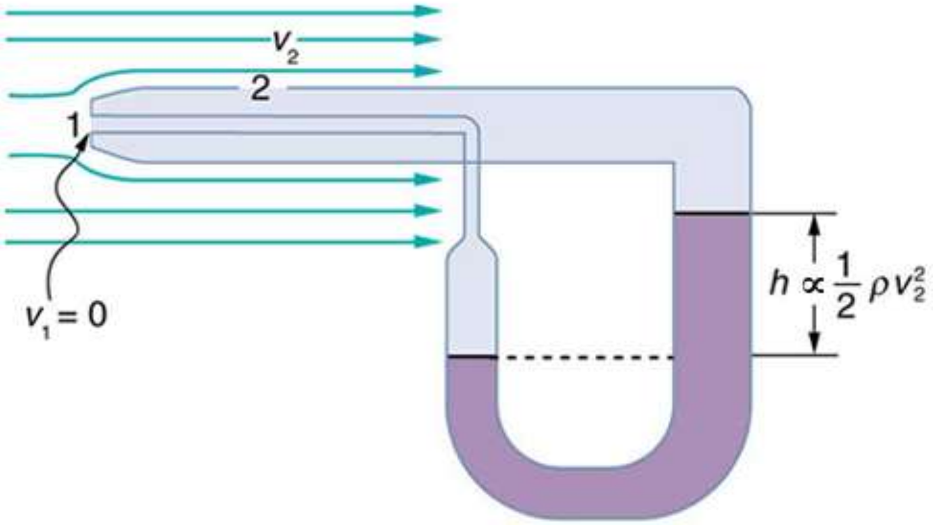
- Sails work the same, except horizontally.





- A manometer can be used to measure velocity.
- The air entering tube 1 has nowhere to go, so it has speed $v_1 = 0$.
- The air at tube 2 just passes by with speed $v_2 > 0$, therefore lower pressure.
- From Bernoulli's equation: $P_1 - P_2 = \frac{1}{2}\rho v_2^2$
- The pressure difference pushes the fluid up a height $h \propto \frac{1}{2}\rho v_2^2$, so $v_2 \propto \sqrt{\frac{2h}{\rho}}$.





12.3 The Most General Applications of Bernoulli’s Equation

- Consider water flowing from the bottom of a dam. Apply Bernoulli's equation:

$$(P_2 - P_1) + \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) = 0$$

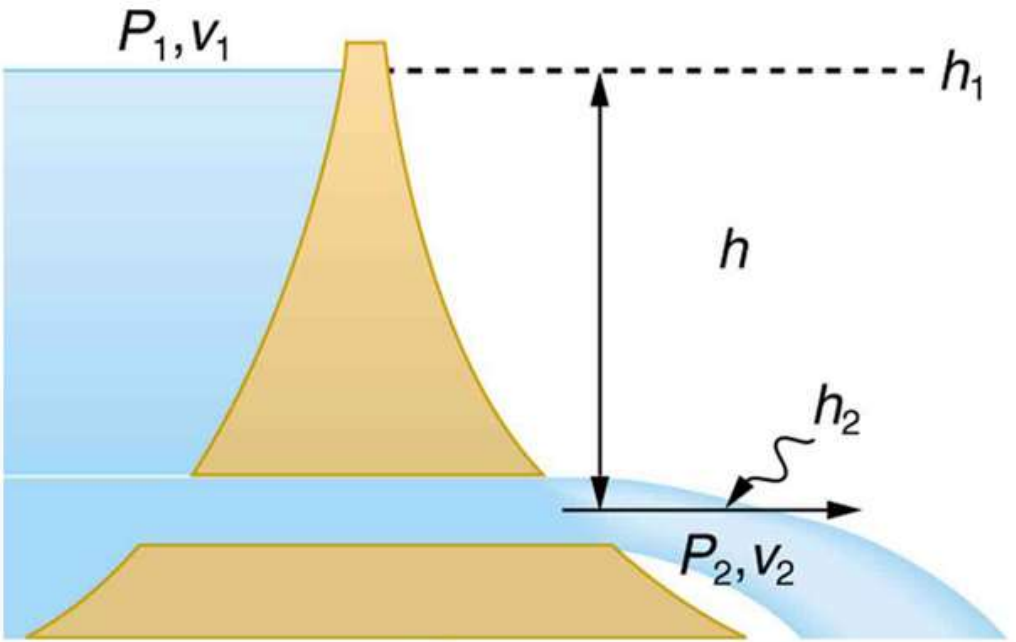
- $P_1 = P_2$ because both must be atmospheric pressure (the fluid is out in the atmosphere, not inside a container):

$$\frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) = 0$$

- Note that ρ cancels. Define $h \equiv h_1 - h_2$ and solve for v_2:

$$v_2^2 = v_1^2 + 2gh$$

- This is simply a kinematic equation for any object falling a distance h. In fluids, it is called Torricelli's theorem.



- Recall that power is energy (or work) per unit time:

$$\bar{P} \equiv \frac{\Delta E}{\Delta t}$$

  - Both power and pressure are denoted P, don't confuse them! I'll use $\bar{P}$ for power to avoid confusion, but it's not standard notation.

- In Bernoulli's equation, the terms are actually energy per unit volume:

$$P + \frac{1}{2}\rho v^2 + \rho g h = \frac{1}{V}\left(PV + \frac{1}{2}mv^2 + mgh\right) = \frac{E}{V} = \text{constant}$$

- Flow rate is $Q \equiv V/t$, so:

$$\bar{P} = \frac{E}{t} = \frac{E}{V}\frac{V}{t} = \left(P + \frac{1}{2}\rho v^2 + \rho g h\right)Q$$

- In other words: multiply Bernoulli's equation by Q to get the power.