PHYS 1P22/92

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Spring 2024

19. Electric Potential and Electric Energy

19.1 Electric Potential Energy: Potential Difference

- **Electric potential energy** is the negative of the work done to bring the charge from infinity:

$$E_p \equiv -W = -\mathbf{F} \cdot \mathbf{r} = -q\mathbf{E} \cdot \mathbf{r}$$

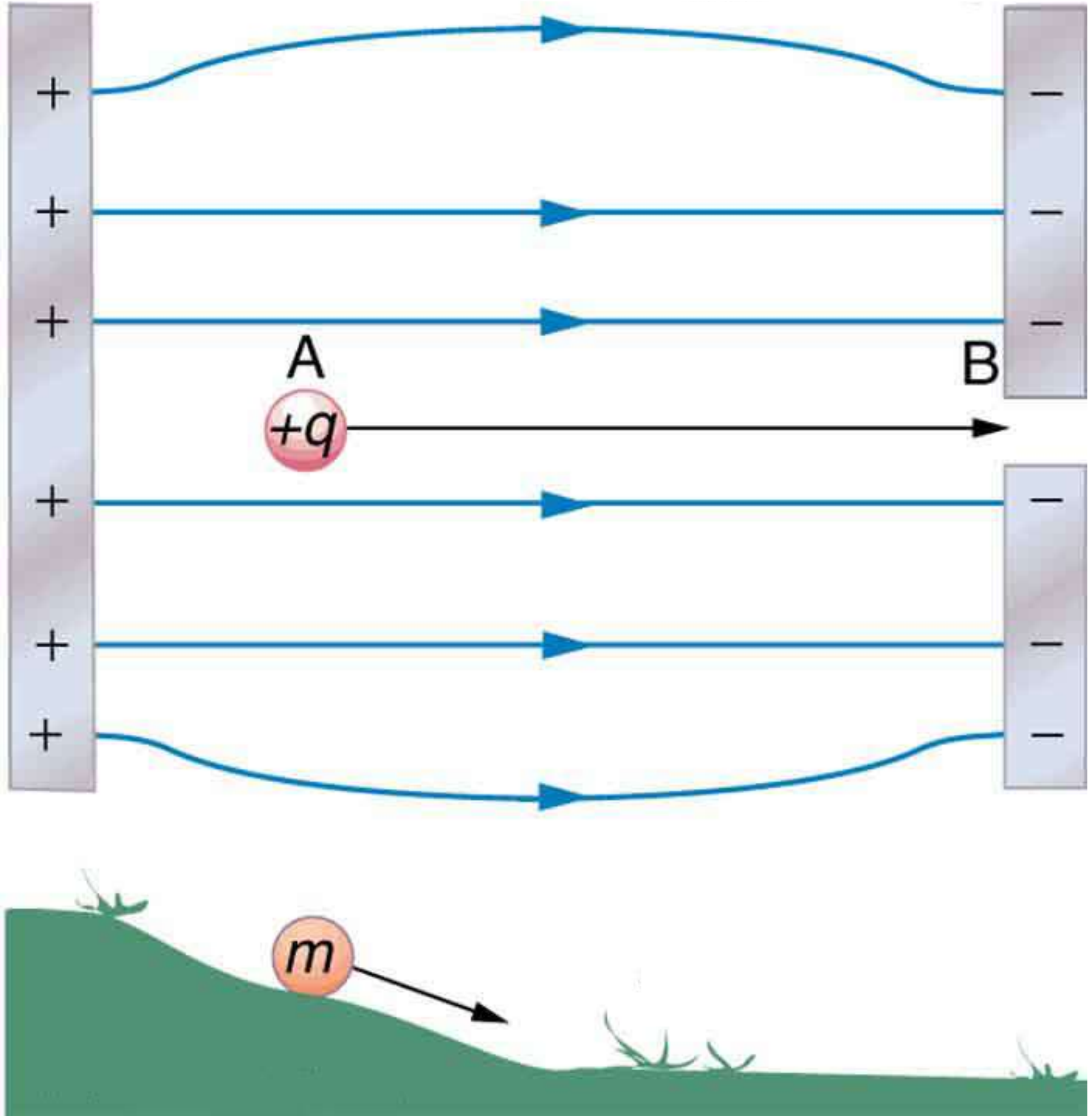
- More precisely (with calculus):

$$E_p = -\int q\mathbf{E} \cdot \mathrm{d}\mathbf{r}$$

- Analogous with gravity, converted to kinetic energy as charge "falls" down electric field.
- **Electric potential** is potential energy per unit charge:

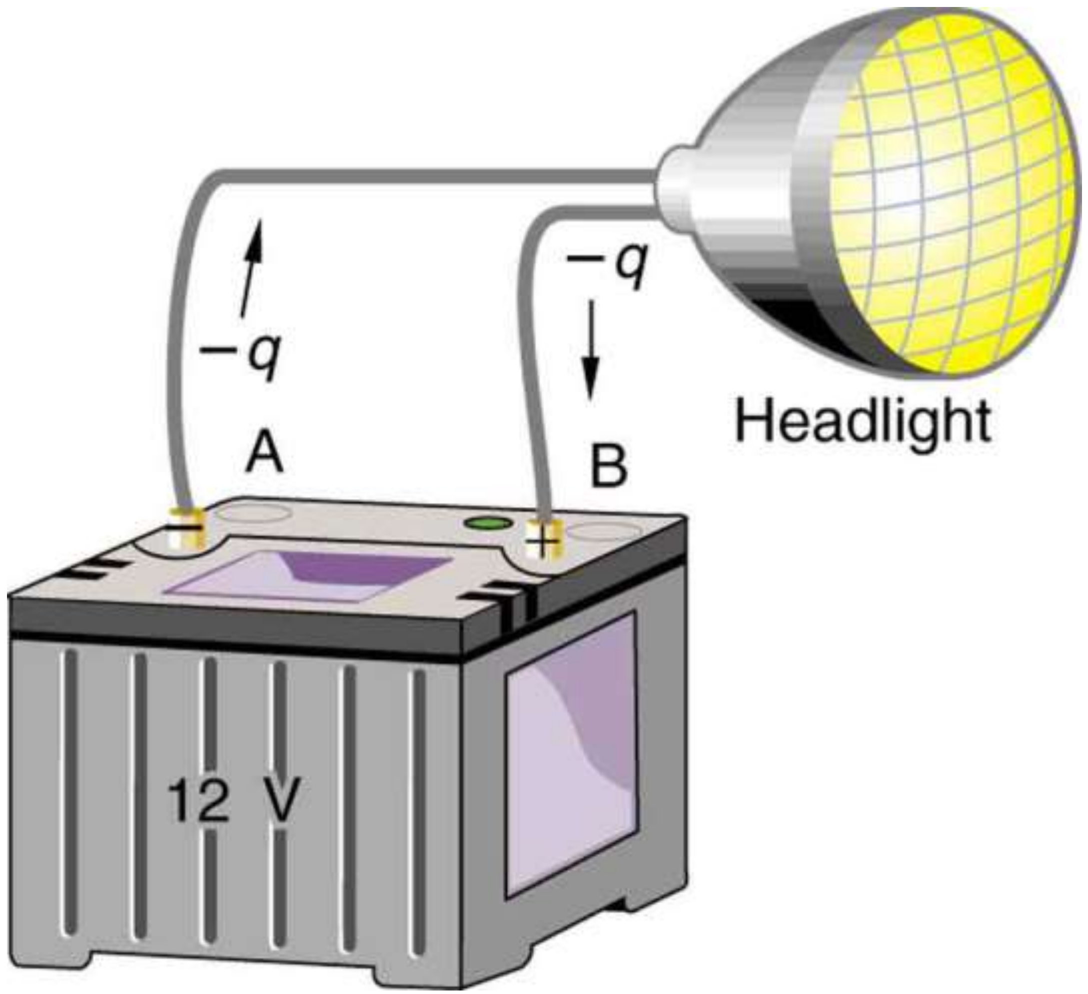
$$V \equiv \frac{E_p}{q}$$

- Units: Volt (V) ≡ J/C.



Voltage

- “Voltage” is the **potential difference** $\Delta V = V_B - V_A$ between two points A and B.
- Voltage is energy per unit charge, so two batteries with same voltage but different charge store different energy.
- Charges going from A to B obtain a change in potential energy $\Delta E_p = q\Delta V$.



Electron volts

- $1\ \text{eV} =$ energy given to elementary charge e accelerated through 1 volt.

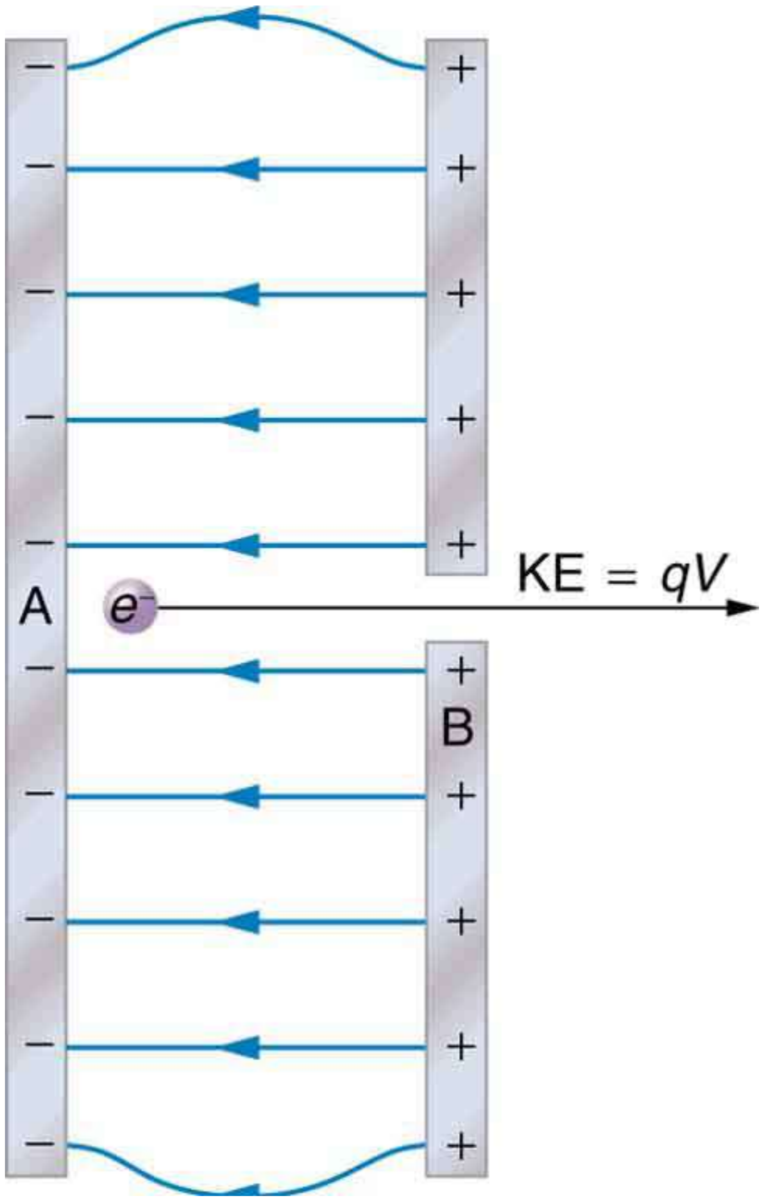
$$\begin{aligned} 1\ \text{eV} &= e \cdot \Delta V \\ &\equiv (1.602\,176\,634 \times 10^{-19}\ \text{C})(1\ \text{V}) \\ &= 1.602\,176\,634 \times 10^{-19}\ \text{J}. \end{aligned}$$

- Can also be used for mass since $E = mc^2$:

$$m = E/c^2$$

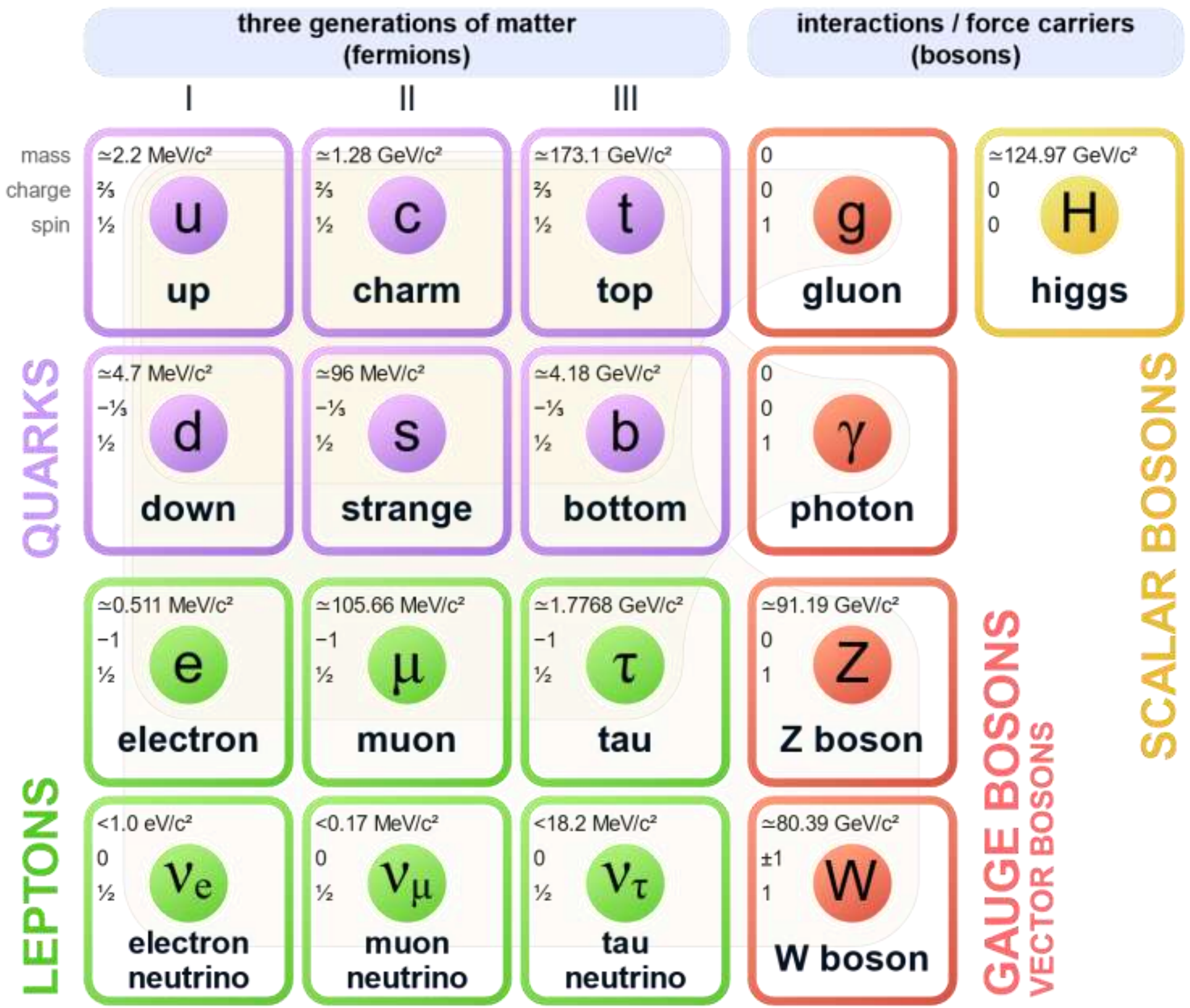
So:

$$\begin{aligned} 1\ \text{eV}/c^2 &\approx (1.6 \times 10^{-19}\ \text{J})/(3.0 \times 10^{8}\ \text{m/s})^2 \\ &\approx 1.8 \times 10^{-36}\ \text{kg} \end{aligned}$$



Standard Model of Elementary Particles

$$\mathrm{keV} = 10^3\ \mathrm{eV}$$
$$\mathrm{MeV} = 10^6\ \mathrm{eV}$$
$$\mathrm{GeV} = 10^9\ \mathrm{eV}$$



19.2 Electric Potential in a Uniform Electric Field

- Work done by electric field to move positive q from A (+) to B (−):

$$W = -\Delta E_p = -q\Delta V$$

- Potential difference:

$$\Delta V = V_B - V_A = -V_{AB}$$

- Definition of work:

$$W = Fd = qEd$$

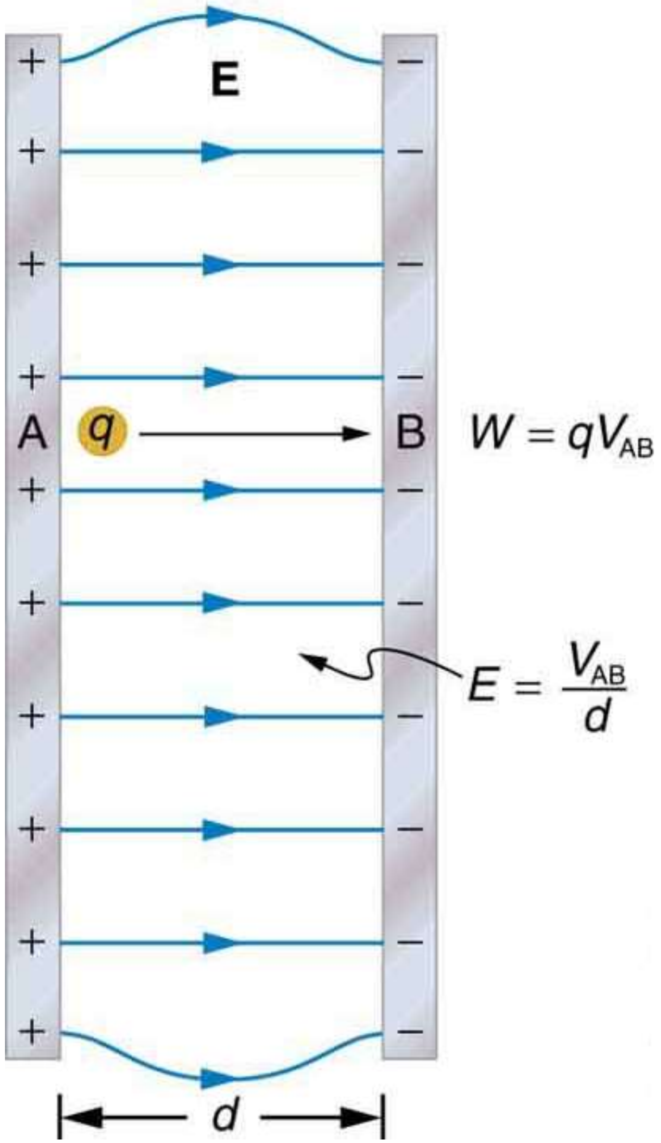
- Compare:

$$qEd = qV_{AB}$$

$$Ed = V_{AB}$$

$$E = \frac{V_{AB}}{d}$$

- Note: units of E are V/m, equivalent to N/C.



Some vector calculus

- Precise definition of electric potential:

$$V \equiv -\int \mathbf{E} \cdot \mathrm{d}\mathbf{r}$$

- Inverting the relation (using the gradient theorem):

$$\mathbf{E} = -\boldsymbol{\nabla} V = -\frac{\mathrm{d}V}{\mathrm{d}\mathbf{r}}$$

- In terms of magnitude:

$$E = -\frac{\mathrm{d}V}{\mathrm{d}r} \approx -\frac{\Delta V}{\Delta r}$$

- Why should you learn calculus? See next slides…

A Mathematical Model for the Determination of Total Area Under Glucose Tolerance and Other Metabolic Curves

MARY M. TAI, MS, EDD

OBJECTIVE — To develop a mathematical model for the determination of total areas under curves from various metabolic studies.

RESEARCH DESIGN AND METHODS — In Tai's Model, the total area under a curve is computed by dividing the area under the curve between two designated values on the X-axis (abscissas) into small segments (rectangles and triangles) whose areas can be accurately calculated from their respective geometrical formulas. The total sum of these individual areas thus represents the total area under the curve.

Tai's Formula Is the Trapezoidal Rule

$$\text{Area} = \frac{1}{2}\sum_{i=1}^{n} x_{i-1}(y_{i-1} + y_i)$$

(Tai's formula)

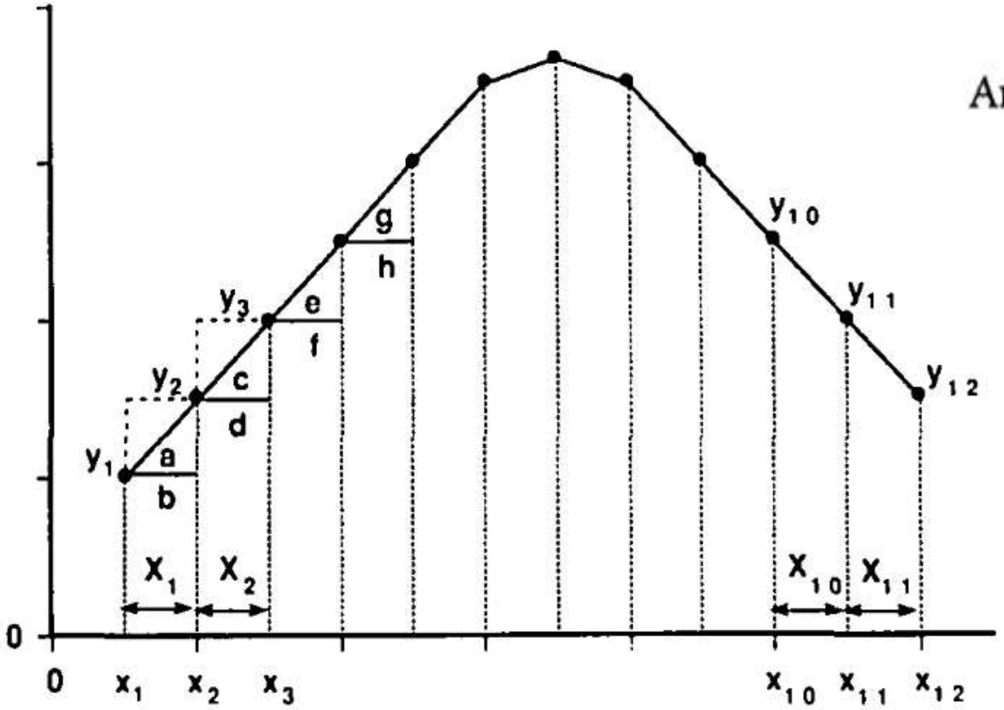


Figure 1—*Total area under the curve is the sum of individual areas of triangles a, c, e, and g and rectangles b, d, f, and h.*

A mathematical model for the determination of total area under glucose tolerance and other metabolic curves

MM Tai - Diabetes care, 1994 - Am Diabetes Assoc

OBJECTIVE To develop a mathematical model for the determination of total areas under curves from various metabolic studies. RESEARCH DESIGN AND METHODS In Tai's Model ...

We were disturbed to read the article by M. M. Tai titled "A Mathematical Model for the Determination of Total Area Under Glucose Tolerance and Other Metabolic Curves" (1). The author seems to claim "Tai's formula" as a new method of computing area under a curve. The formula given is simply the trapezoidal rule, published in many beginning calculus texts (for example, see Swokowski [2] or Faires and Faires [3]). Although we do not have a first reference, it is our understanding that the trapezoidal rule was known to Isaac Newton in the 17th century. Further, her article omitted any reference to the magnitude of error of the area approximation when the true curve is unknown, as is the case for measuring glucose tolerance.

(This is actually just $\int f(x)\,dx$)

19.3 Electrical Potential Due to a Point Charge

- For a point charge Q:

$$V = k\frac{Q}{r}$$

- $k \approx 8.987\,551\,792\,3(14) \times 10^9\ \mathrm{N \cdot m^2/C^2}$ is Coulomb's constant.
- Only potential difference matters (like gravity, e.g. choosing the ground to be zero potential). In this case we choose to have zero potential at infinity:

$$\lim_{r\to\infty} V = "\frac{1}{\infty}" = 0.$$

- Electric field:

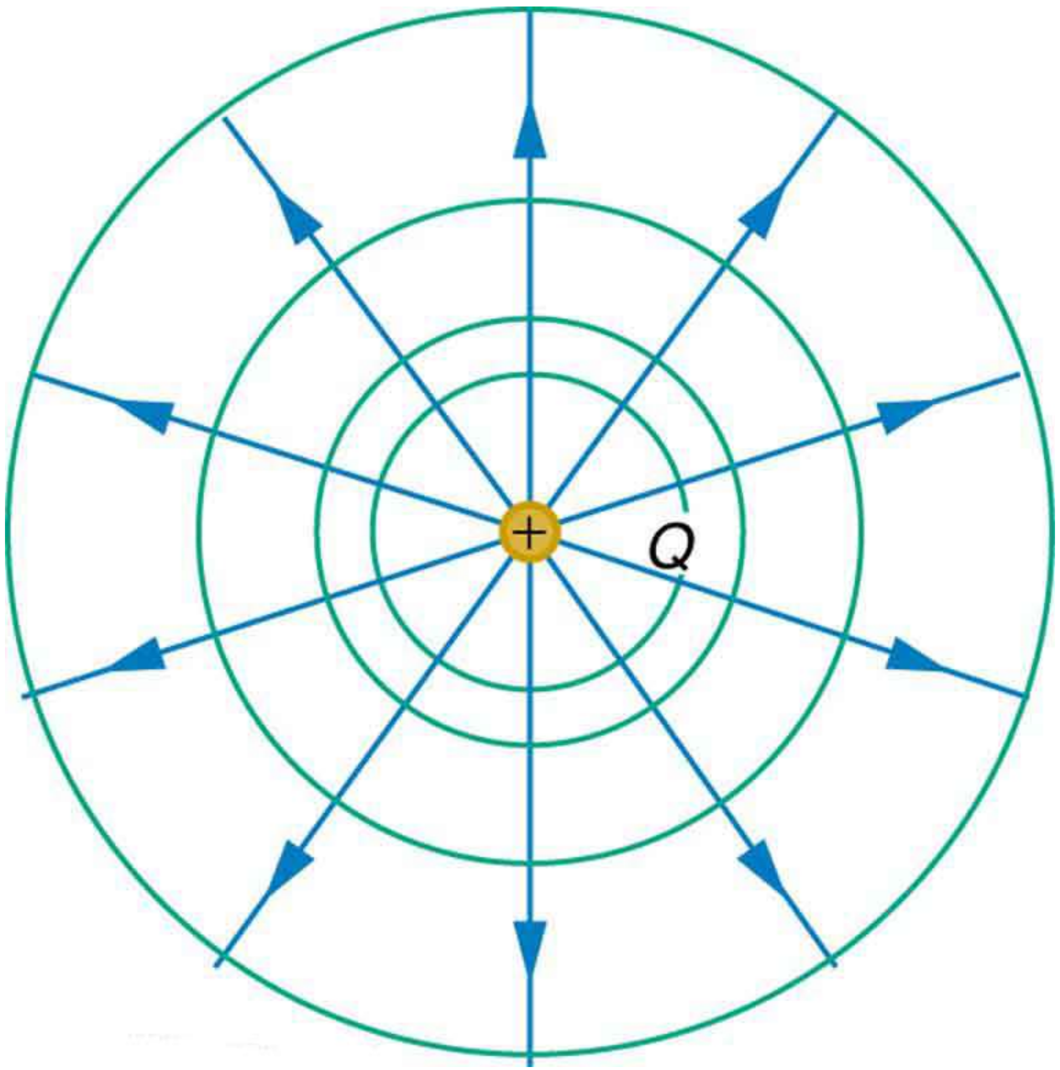
$$|\mathbf{E}| = \frac{\mathrm{d}V}{\mathrm{d}r} = k\frac{Q}{r^2}$$

- And we find Coulomb's law, since $\mathbf{F} = q\mathbf{E}$:

$$|\mathbf{F}| = k\frac{Qq}{r^2}$$

19.4 Equipotential Lines

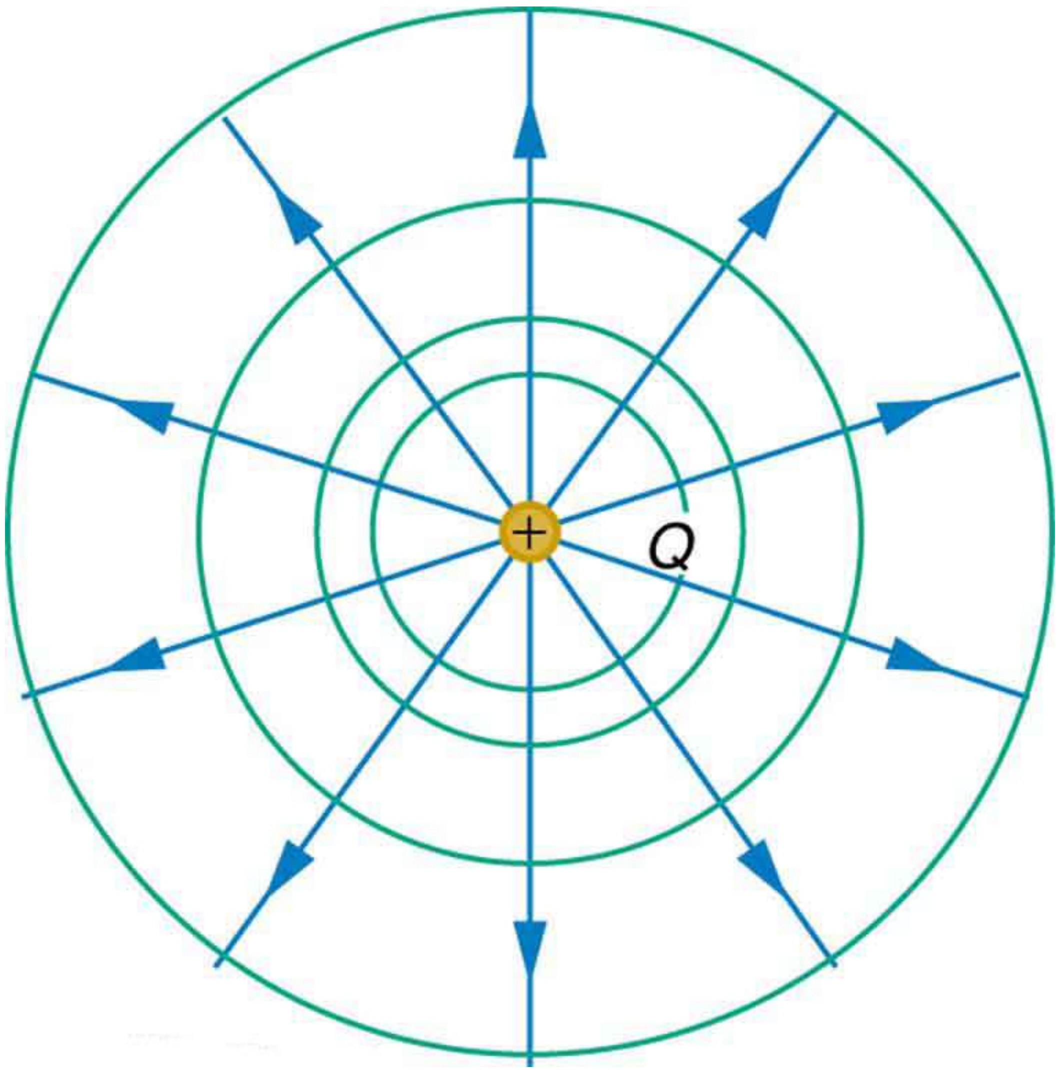
- Equipotential = line of equal potential (green).
- No work required to move along an equipotential line, since $W = -q\Delta V$ but $\Delta V = 0$.
- Work **is** required to move between different lines!

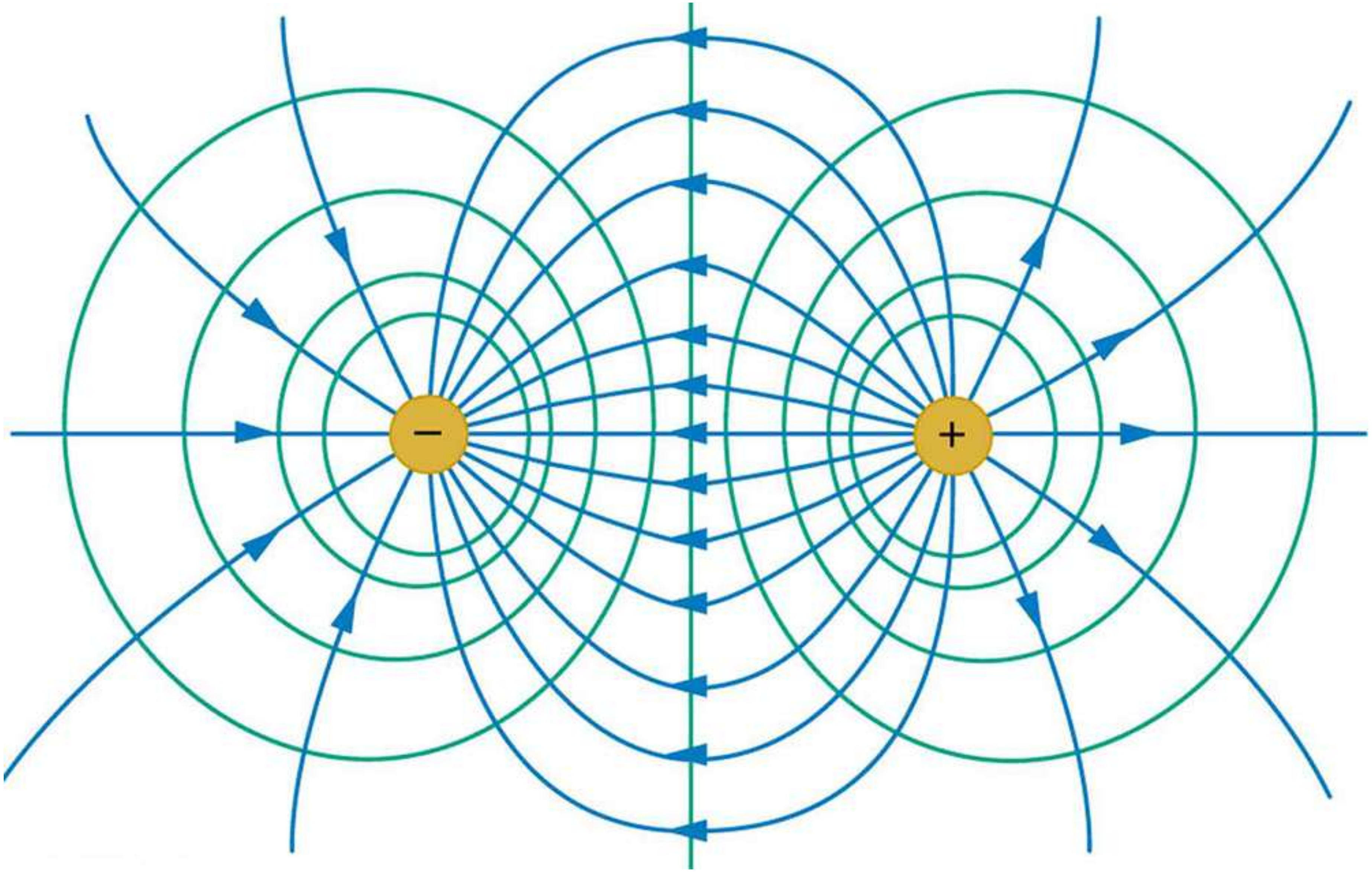


- Equipotential lines (green) are always perpendicular to electric field lines (blue), because
  $W \equiv \mathbf{F} \cdot \mathbf{d} \equiv Fd \cos\theta = qEd \cos\theta = 0$

If $q, E, d \neq 0$ then $\cos\theta = 0$, therefore $\theta = 90°$.

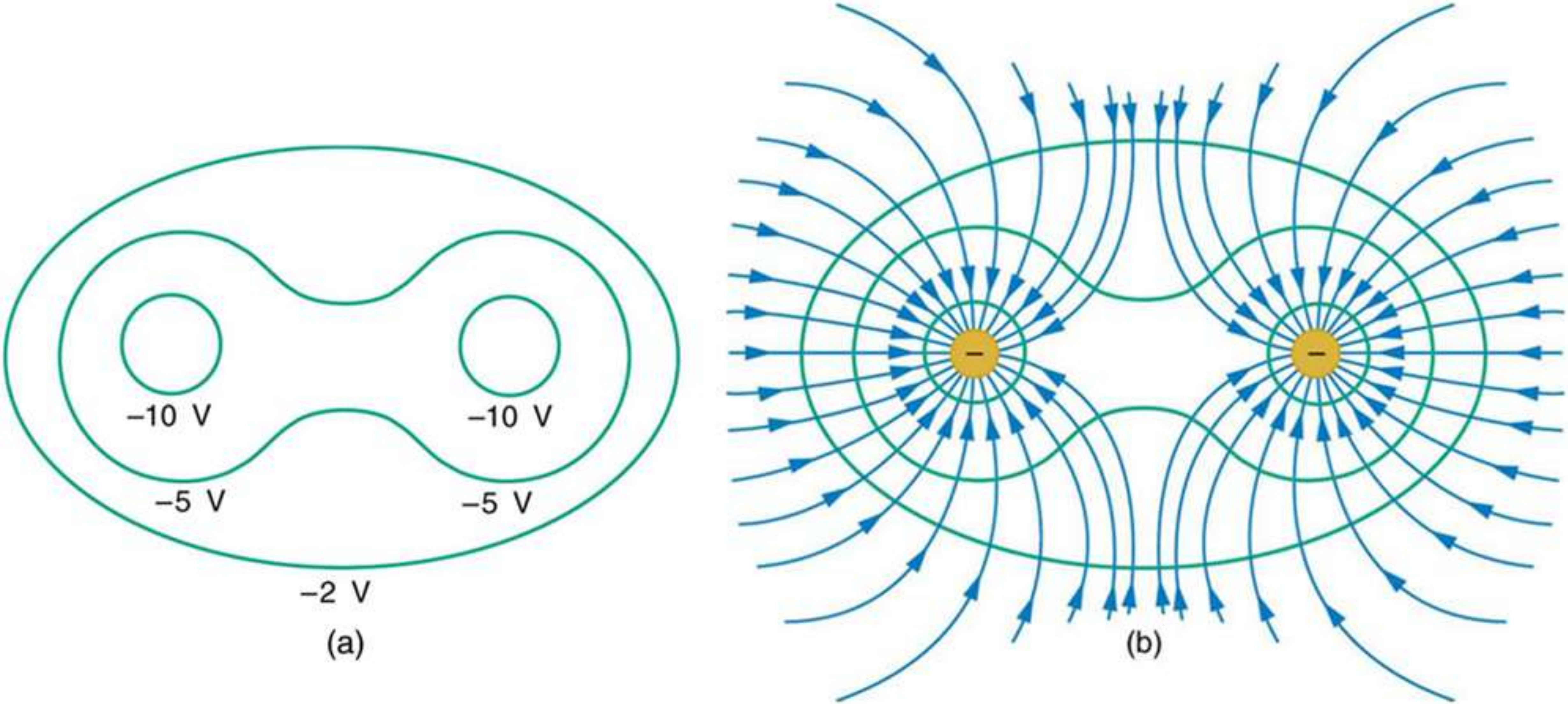
- In a conductor, **E** is perpendicular to the surface. Thus a conductor is an equipotential surface.
- Conductor can be fixed to 0 voltage by grounding (connecting to the Earth).



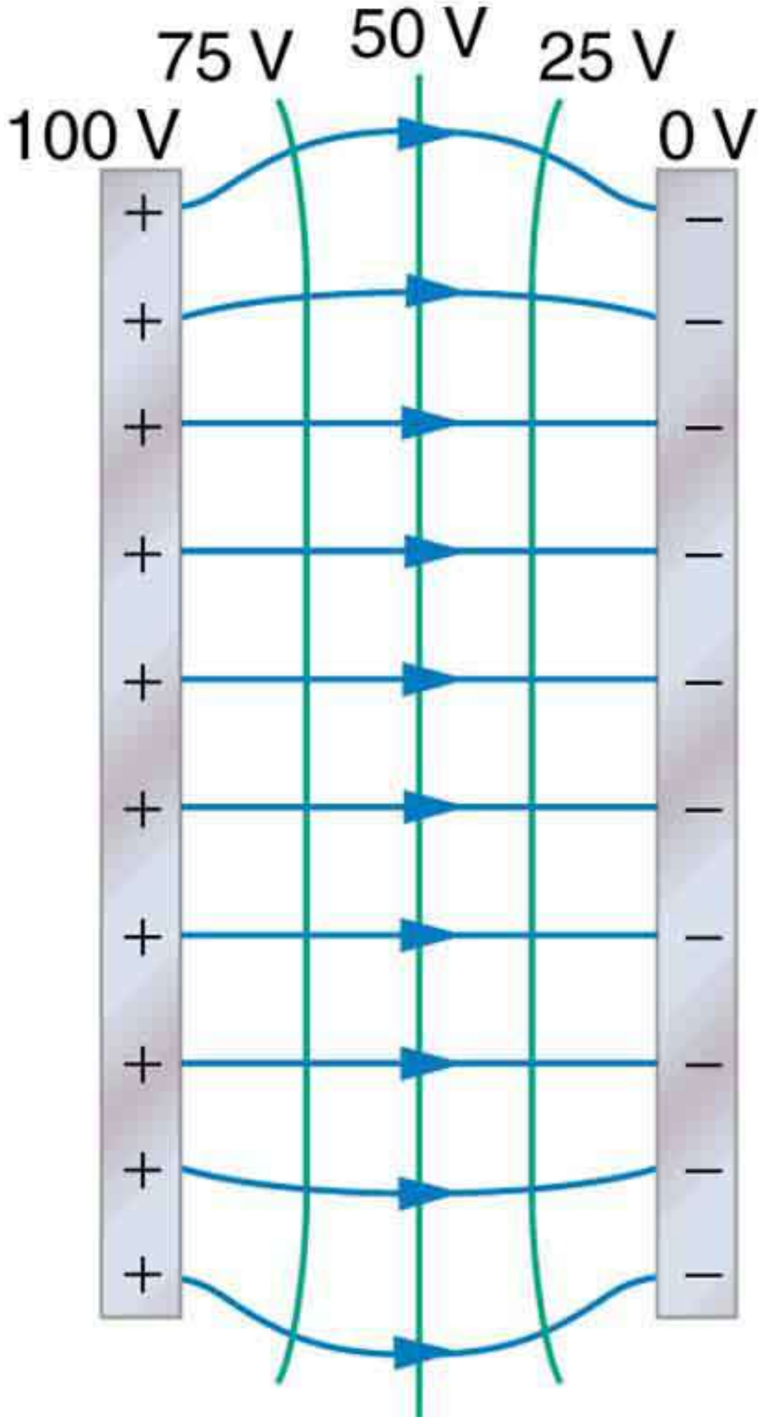


Note:

1. Equipotential lines are perpendicular to electric field lines. This can be used to draw one if the other is known.
2. Potential is most positive near the positive charge, most negative near the negative charge.



(a) Measure potential in lab and draw lines, (b) add perpendicular electric field lines.


75 V
50 V
25 V
100 V
0 V

Notes on equipotentials

- In 2D, the equipotentials are lines. In 3D, they are **surfaces**.
- Equipotential line density doesn't indicate magnitude (like field lines). It indicates **rate of change**.

Demonstration

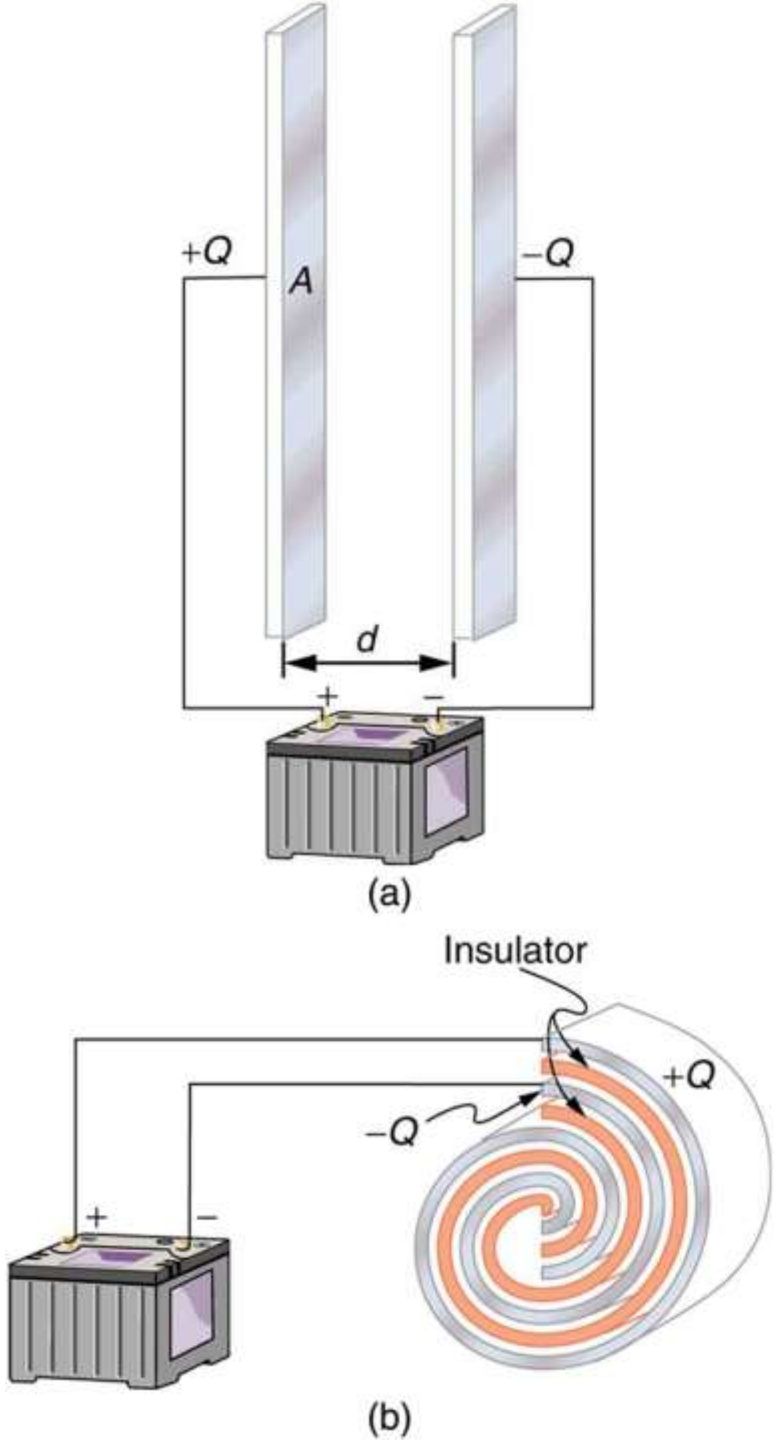
Simulations of electric field lines and equipotential lines:

https://phet.colorado.edu/sims/html/charges-and-fields/latest/charges-and-fields_all.html

https://prancer.physics.louisville.edu/modules/electric_field/index.html

19.5 Capacitors and Dielectrics

- A capacitor is a device used to store electric charge.
- Usually: two conductors close to each other but not touching.
- (a) air, (b) insulator.
- Initially uncharged. After connecting battery, charges $+Q$ and $-Q$ are added and we say the capacitor stores a charge Q.
- Q depends on voltage and physical characteristics.
- Capacitor is neutral overall.



- Parallel plate capacitor: each pair of charges connected by one field line, so $E \propto Q$.
- $V = Ed$, so $V \propto E$.
- Therefore:

$$Q \propto V$$

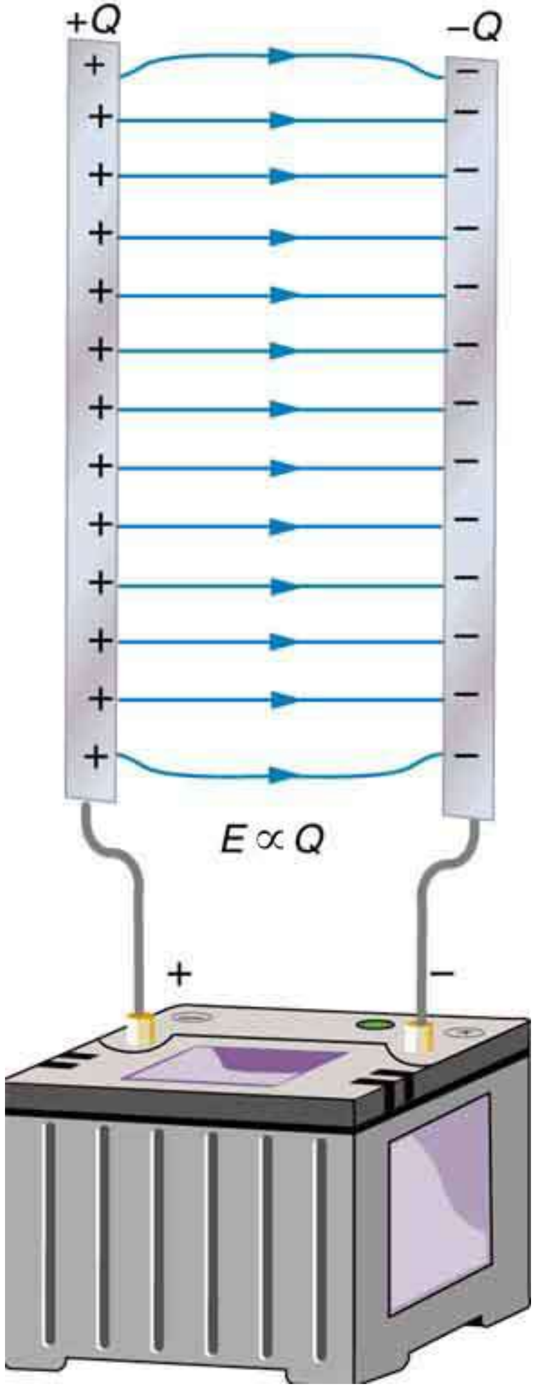
- The proportionality constant is the capacitance C:

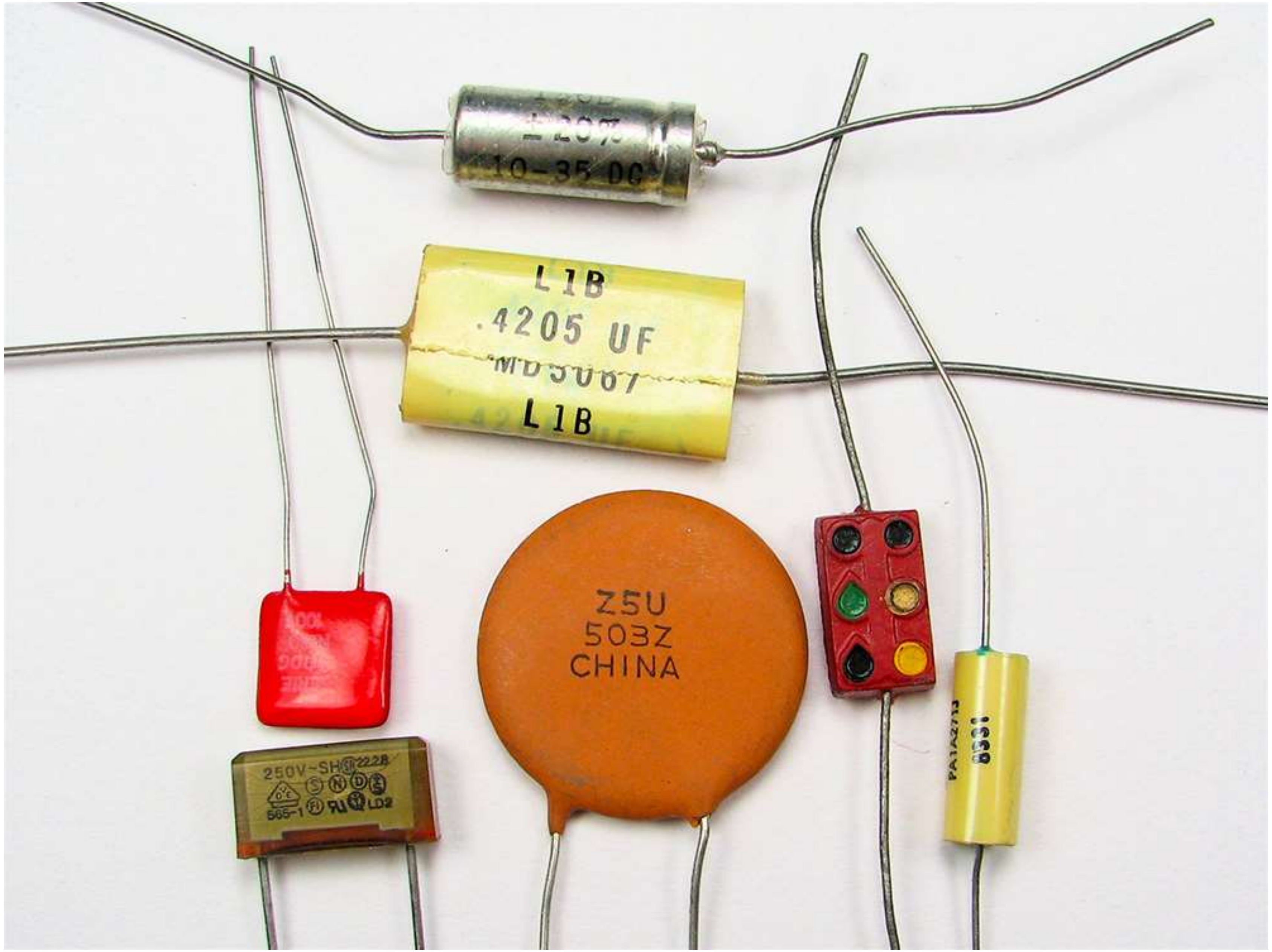
$$Q = CV$$

or

$$C = \frac{Q}{V}$$

- Units: farad (F) $\equiv$ C/V $= \mathrm{kg}^{-1} \cdot \mathrm{m}^{-2} \cdot \mathrm{s}^{4} \cdot \mathrm{A}^{2}$.
- (Notice the font: C is capacitance, C is coulomb, V is voltage, V is volt!)
- 1 coulomb is a lot, so 1 farad is also a lot. Typically C is measured in picofarads pF $\equiv 10^{-12}$ F up to millifarads mF $\equiv 10^{-3}$ F.




10-35 DC
L1B
.4205 UF
L1B
Z5U
503Z
CHINA
250V~SH

- Increase A: more charge. So $C \propto A$.
- Decrease d: force increases ($\mathrm{F} \propto 1/d^2$), pulls more charges in. So $C \propto 1/d$.
- Thus

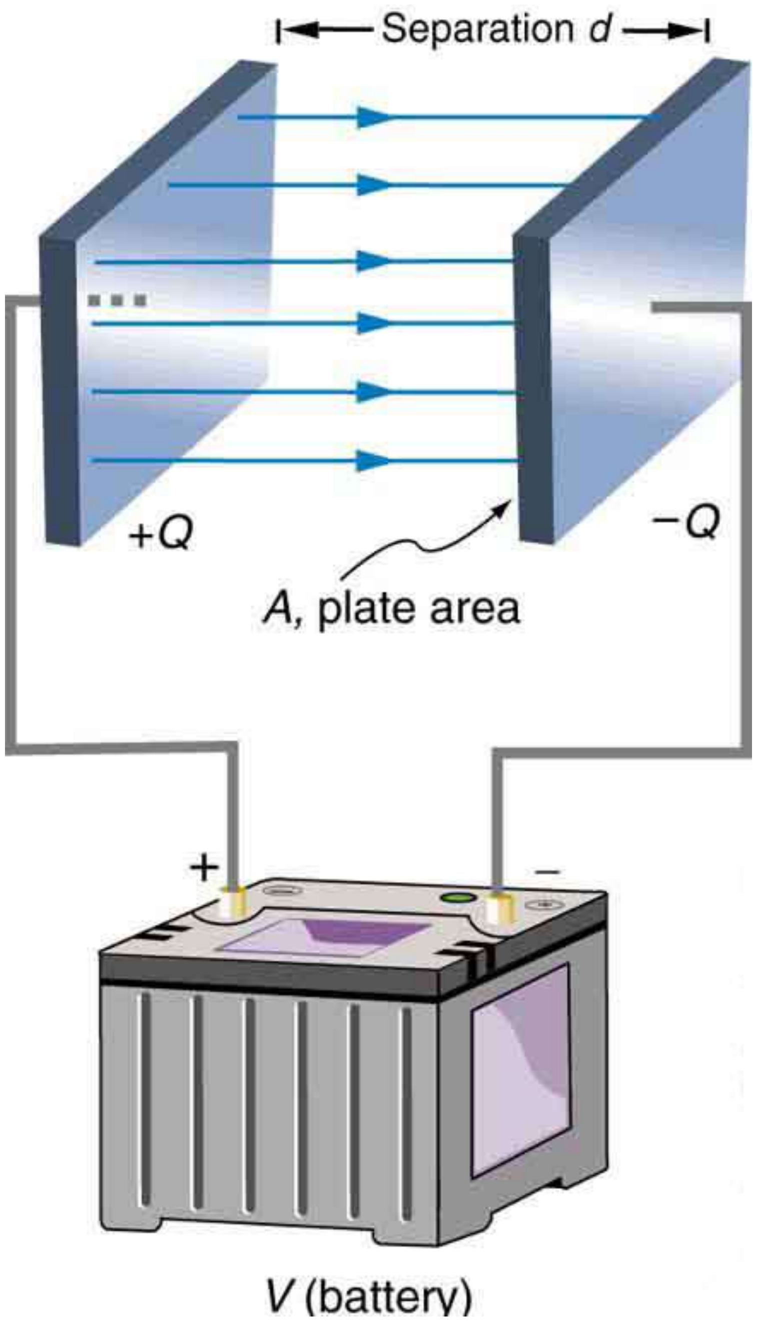
$$C \propto \frac{A}{d}.$$

- Label the proportionality constant:

$$C = \varepsilon_0 \frac{A}{d}$$

- ε_0 is called "vacuum permittivity" or "permittivity of free space".

$$\varepsilon_0 \approx 8.854\,187\,8188(14) \times 10^{-12}\ \mathrm{F} \cdot \mathrm{m}^{-1}$$



Dielectrics

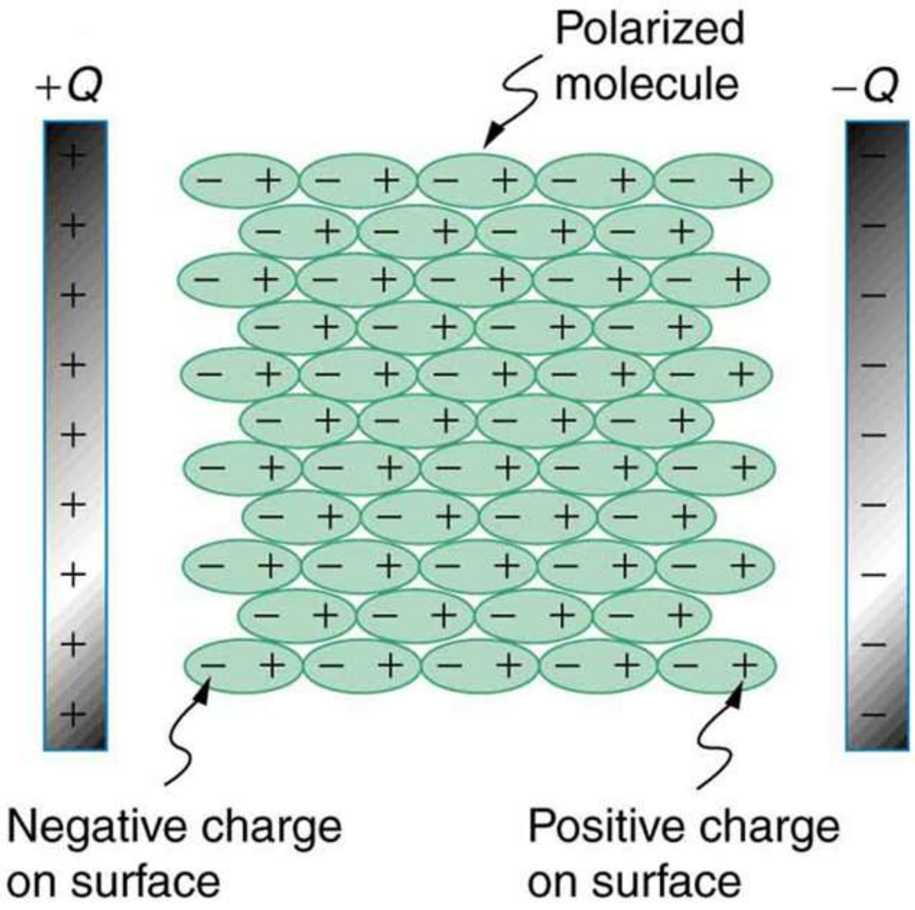
- With small d, electric field increases since $E = V/d$.
- When E is large enough, electrical breakdown occurs: an insulator (including air or even vacuum) becomes a conductor and the capacitor doesn't work anymore.
- We can add a dielectric, an insulator that can withstand greater E.

$$C = \kappa \varepsilon_0 \frac{A}{d}$$

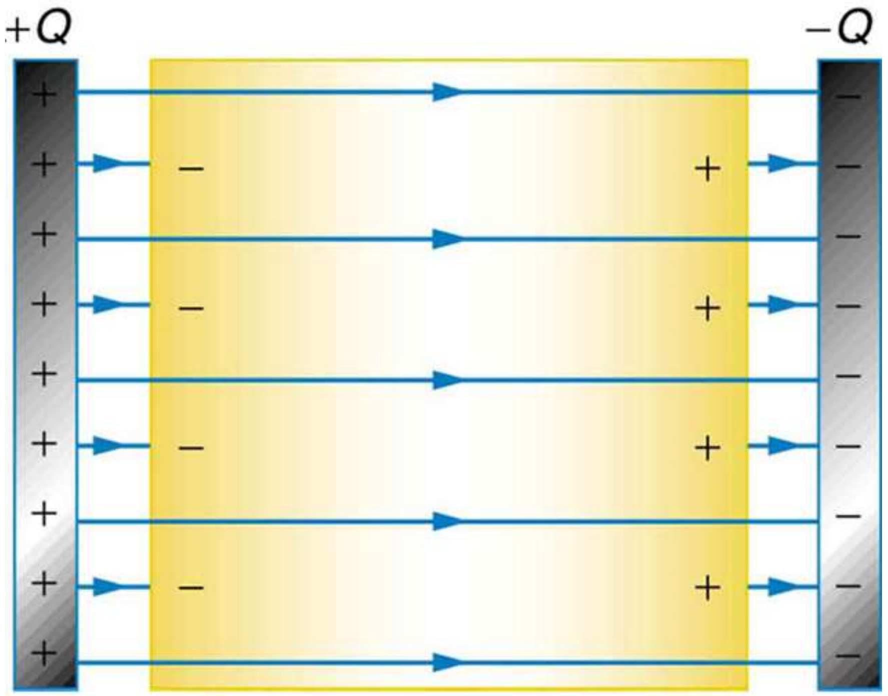
- $\kappa =$ **dielectric constant** of the specific material.
- **Dielectric strength** = maximum E before breakdown.

Material	Dielectric constant κ	Dielectric strength (10^6 V/m)
Vacuum	1.00000	—
Air	1.00059	3
Bakelite	4.9	24
Fused quartz	3.78	8
Neoprene rubber	6.7	12
Nylon	3.4	14
Paper	3.7	16
Polystyrene	2.56	24
Pyrex glass	5.6	14
Silicon oil	2.5	15
Strontium titanate	233	8
Teflon	2.1	60
Water	80	—

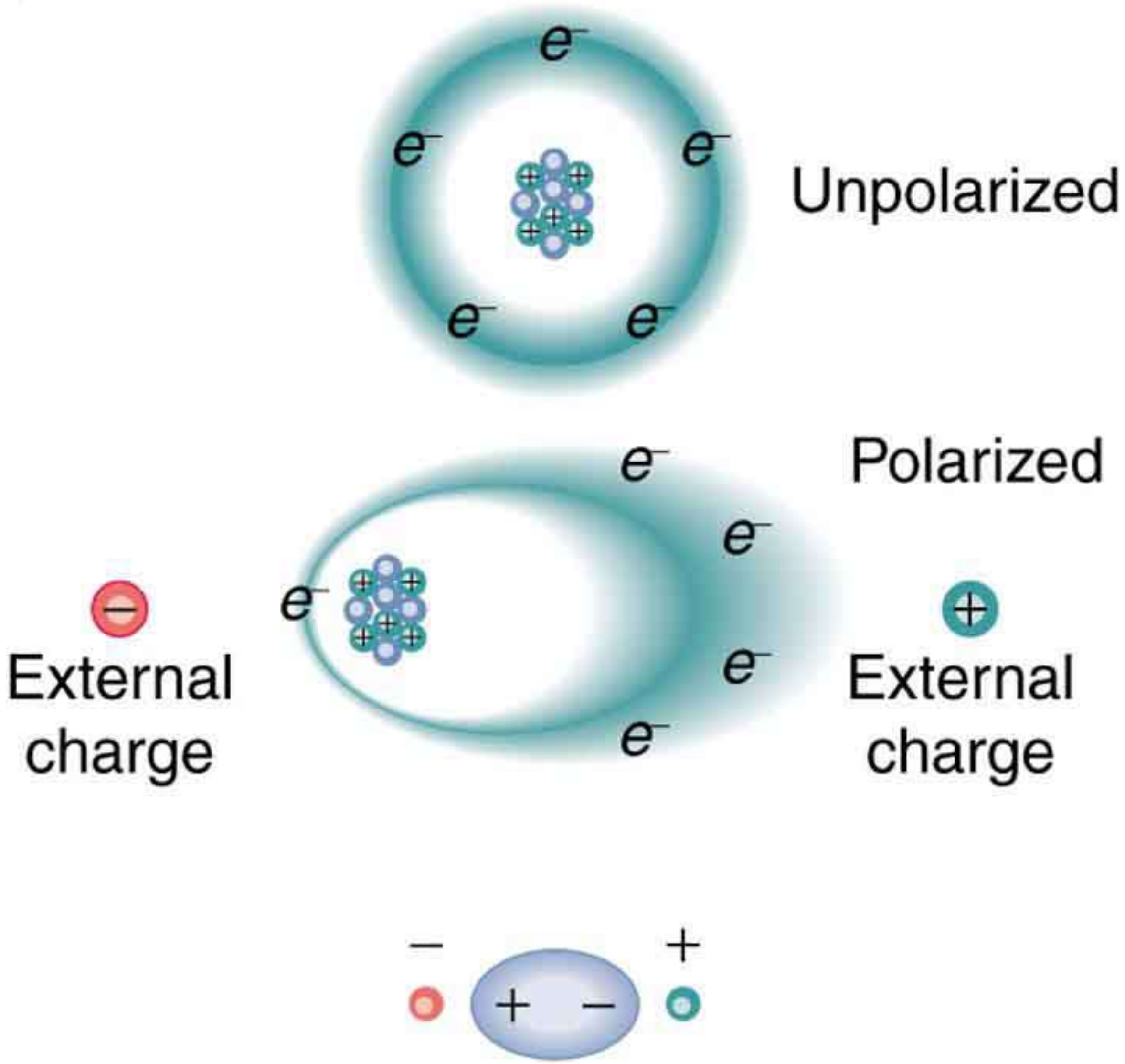
How do dielectrics work?



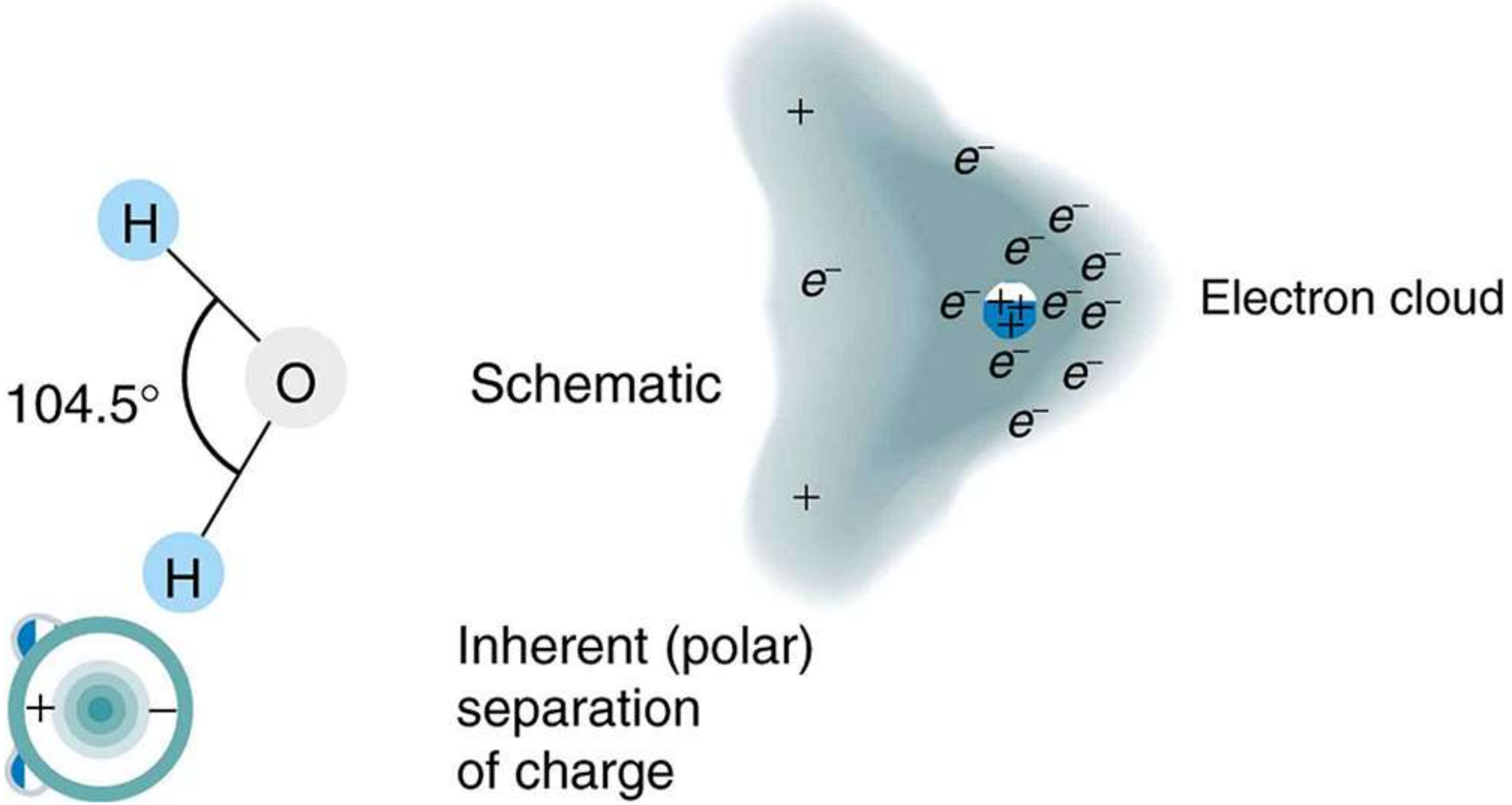
Polarization attracts more charge, increasing C.



E is reduced (field lines end on surface).
$V = Ed$ decreases, so $C = Q/V$ increases.
$\kappa \equiv E_{\text{vacuum}}/E_{\text{dielectric}}$.



Large-scale view of polarized atom



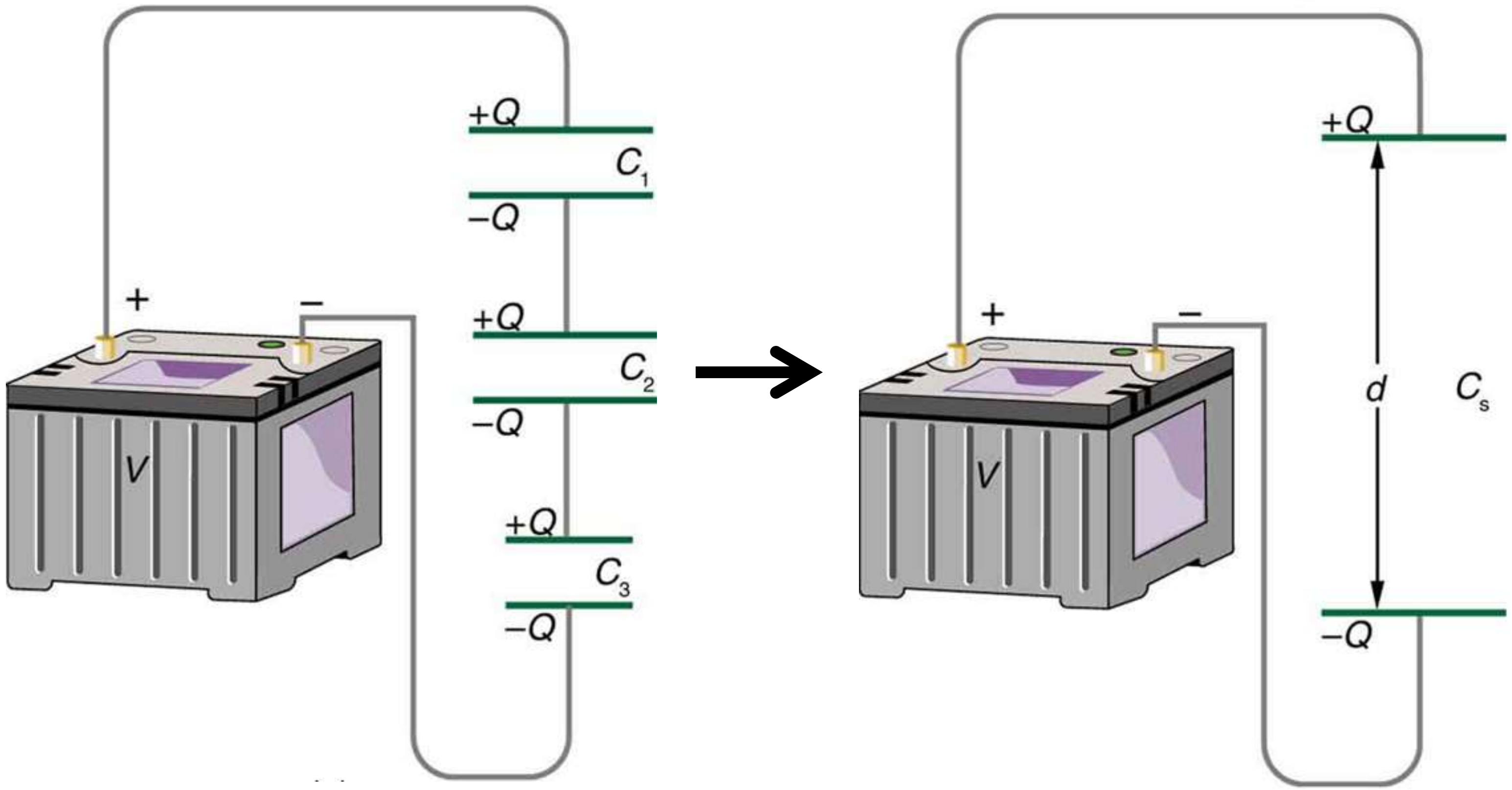
Water molecule: electrons concentrated around oxygen atom.

Capacitor demonstration

https://phet.colorado.edu/sims/html/capacitor-lab-basics/latest/capacitor-lab-basics_en.html

19.6 Capacitors in Series and Parallel

Series



$$V = \frac{Q}{C}$$

$$\downarrow$$

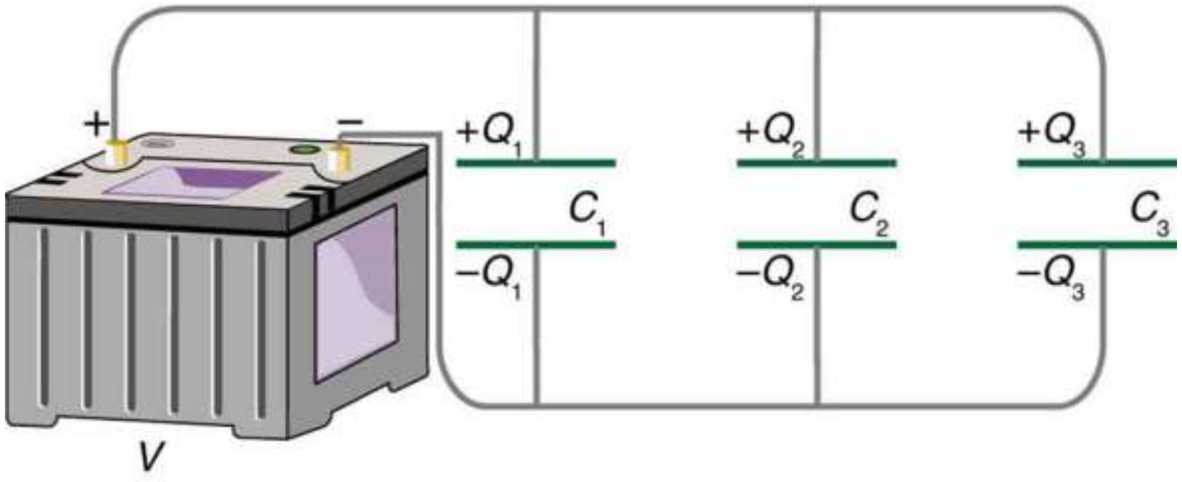
$$V_S = V_1 + V_2 + V_3 = \sum_i V_i$$

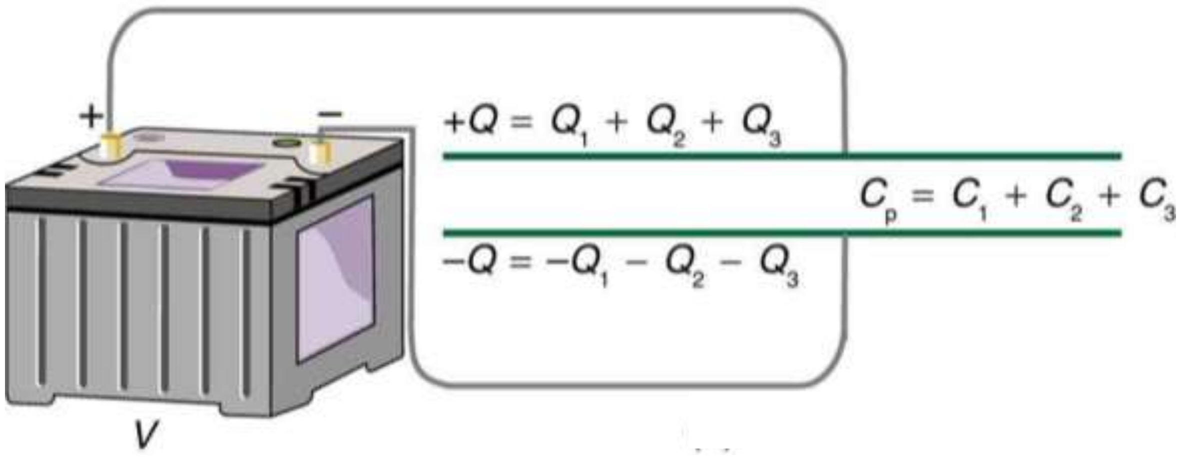
$$\downarrow$$

$$\frac{Q}{C_S} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} = Q\sum_i \frac{1}{C_i} \rightarrow \frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \sum_i \frac{1}{C_i}$$

$$C_S = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} = \frac{1}{\sum_i \frac{1}{C_i}}$$

Parallel





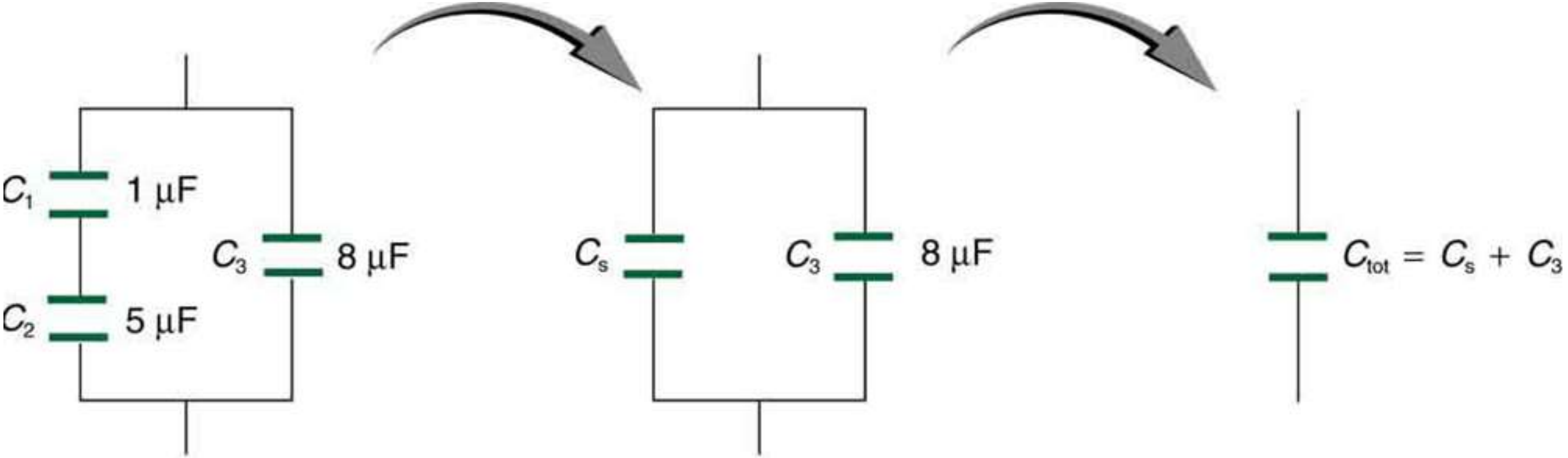
$$Q = CV$$

$$Q_P = Q_1 + Q_2 + Q_3 = \sum_i Q_i$$

$$C_P V = C_1 V + C_2 V + C_3 V = V \sum_i C_i$$

$$C_P = C_1 + C_2 + C_3 = \sum_i C_i$$

Both series and parallel



C_1 and C_2 are in series: $C_S = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}}$

C_S and C_3 are in parallel

19.7 Energy Stored in Capacitors

- Energy in capacitor = potential energy = work done to charge it.
- From calculus:

$$E = \int V \, \mathrm{d}Q = \int \frac{Q}{C} \mathrm{d}Q = \frac{1}{2}\frac{Q^2}{C}$$

- Alternative expressions using $Q = CV$:

$$E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$