PHY 256 Lecture 2

$z = a + ib$

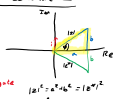
$z \in \mathbb{C}$

$a, b \in \mathbb{R}$

$i \to -i$

complex conjugate

$z^* = a - ib$



$z = re^{i\phi}$

$r = |z|$

Euler's formula

$e^{i\phi} = \cos\phi + i\sin\phi$

$|z|^2 = a^2 + b^2 = (a+ib)^2$

$|z|^2 = zz^*$

$(a+ib)(a-ib) = a^2 + b^2$

$|e^{i\phi}| = 1$

$|re^{i\phi}| = r = |z|$

Linear algebra

Dirac notation
Bra-ket notation

$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} \in \mathbb{C}^n$

$|A\rangle$ $|\psi\rangle$ $|\beta\rangle$

$|\psi\rangle$ $|\text{Bob}\rangle$

$|\text{Cat is alive}\rangle$

$|\psi\rangle = \begin{pmatrix} a \\ c \end{pmatrix}$ $a, \psi_2 \in \mathbb{C}$

$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ $a, b \in \mathbb{C}$

addition $|\psi\rangle + |\phi\rangle = \begin{pmatrix} \psi_1 + \phi_1 \\ \psi_2 + \phi_2 \end{pmatrix}$

$\lambda \in \mathbb{C}$

mult by scalar $\lambda|\psi\rangle = \begin{pmatrix} \lambda\psi_1 \\ \lambda\psi_2 \end{pmatrix}$

$\lambda(|\psi\rangle + |\phi\rangle) = \lambda|\psi\rangle + \lambda|\phi\rangle$

$(\alpha + \beta)|\psi\rangle = \alpha|\psi\rangle + \beta|\psi\rangle$

$\alpha, \beta \in \mathbb{C}$

Dual vectors

ket $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

bra $\langle\psi| = (\psi_1^* \;\; \psi_2^*)$

$|\psi\rangle \neq \langle\psi|$ $\left(\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \neq (\psi_1^* \;\; \psi_2^*)\right)$

Inner product

$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \langle\psi| = (\psi_1^* \;\; \psi_2^*)$

$|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$\langle\psi|\phi\rangle = (\psi_1^* \;\; \psi_2^*)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \psi_1^*\phi_1 + \psi_2^*\phi_2$

bra(c)ket

$\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

Norm

$\|\psi\|^2 = \langle\psi|\psi\rangle = (\psi_1^* \;\; \psi_2^*)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_1^*\psi_1 + \psi_2^*\psi_2$

$\|\psi\| = \sqrt{\langle\psi|\psi\rangle} = \sqrt{|\psi_1|^2 + |\psi_2|^2}$

Hilbert space

$\mathbb{C}^n$

- $\|\psi\| \geq 0$
- $\langle\psi|\psi\rangle = \langle\psi|\psi\rangle^*$
- $\langle\psi|\left(\alpha|\psi\rangle + \beta|\phi\rangle\right) = \alpha\langle\psi|\psi\rangle + \beta\langle\psi|\phi\rangle$

$\alpha, \beta \in \mathbb{C}$

$|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$

(+ complete metric space)

$\langle\psi|\phi\rangle \in \mathbb{C}$

Bases (orthonormal)

$\mathbb{C}^n$ $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, \ldots |\psi_n\rangle$

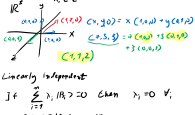
$|\beta_i\rangle$ $i = 1, \ldots n$

1. Span $\mathbb{C}^n$

$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |\beta_i\rangle = \lambda_1|\beta_1\rangle + \cdots + \lambda_n|\beta_n\rangle$

$\lambda_i \in \mathbb{C}$

$\mathbb{R}^2$



2. Linearly independent

If $\sum_{i=1}^{n} \lambda_i |\beta_i\rangle = 0$ then $\lambda_i = 0 \;\forall i$

$2|\beta_2\rangle + 5|\beta_8\rangle = |\beta_3\rangle$

3. Orthogonal

$\langle\beta_i|\beta_j\rangle = 0 \quad \forall i \neq j$

4. Unit norm

$\|\beta_i\| = \sqrt{\langle\beta_i|\beta_i\rangle} = 1 \quad \forall i$

3+4. Orthonormal

$\langle\beta_i|\beta_j\rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

kronecker delta — Kronecker

$\mathbb{C}^2$ 1. Span $|\psi\rangle = \lambda_1|\beta_1\rangle + \lambda_2|\beta_2\rangle$

$\lambda_1, \lambda_2 \in \mathbb{C}$

2. L.I. $|\beta_1\rangle \neq \lambda|\beta_2\rangle$

3+4. O.N. $\langle\beta_1|\beta_2\rangle = 0$

$\|\beta_1\| = \|\beta_2\| = 1$

Standard Basis

$|1,\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1,\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Matrices

$n \times n$ $n = 2$

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $A_{11}, A_{12}, A_{21}, A_{22} \in \mathbb{C}$

$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ ~~$|\psi\rangle A$~~

$A|\psi\rangle = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A_{11}\psi_1 + A_{12}\psi_2 \\ A_{21}\psi_1 + A_{22}\psi_2 \end{pmatrix}$

~~$A\langle\psi|$~~

$\langle\psi|A = (\psi_1^* \;\; \psi_2^*)\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) = (\psi_1^* A_{11} + \psi_2^* A_{21} \;/\; \psi_1^* A_{12} + \psi_2^* A_{22})$

Adjoint

conjugate transpose

$A^\dagger = \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix}$

$|\psi\rangle^\dagger = \langle\psi|$ $(A|\psi\rangle)^\dagger = \langle\psi|A^\dagger$

$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbb{1}|\psi\rangle = |\psi\rangle$ $\langle\psi|\mathbb{1} = \langle\psi|$

involution

$(A^\dagger)^\dagger = A$ $(z^*)^* = z$

$A\left(\alpha|\psi\rangle + \beta|\phi\rangle\right) = \alpha A|\psi\rangle + \beta A|\phi\rangle$

Outer Product

inner: bra ket → number
outer: ket bra → matrix

matrix z m^2
vector $?$ m^1
scalar 0 $?=m^0$

$\langle\psi| = \langle\psi^* \;\; \psi^*|$ $|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$\langle\psi|\phi\rangle$ inner $|\phi\rangle\langle\psi| = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}(\psi_1^* \;\; \psi_2^*) = \begin{pmatrix} \phi_1\psi_1^* & \phi_1\psi_2^* \\ \phi_2\psi_1^* & \phi_2\psi_2^* \end{pmatrix}$

The Completeness Relation

$\langle\beta_j|\psi\rangle = \sum_{i=1}^{n} \lambda_i \langle\beta_j|\beta_i\rangle$ (λ: sum over i = summation var)

$\lambda_i \in \mathbb{C}$

$|\beta_1\rangle\, |\beta_2\rangle \cdots |\beta_n\rangle$

$\langle\beta_i|^\dagger = |\beta_i\rangle$

$\langle\beta_i|\beta_j\rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$\langle\beta_j|\psi\rangle = \sum_{i=1}^{n} \lambda_i \delta_{ij} = \begin{cases} 0 + \cdots + 0 & i = 1 \\ \lambda_j & i = j \\ 0 + \cdots + 0 & i = n \end{cases}$

$= \lambda_j$

$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |\beta_i\rangle = \sum_{i=1}^{n} \underbrace{\langle\beta_i|\psi\rangle}_{\text{scalar}} |\beta_i\rangle$ (sum over i; scalar; vector) $\neq |\psi\rangle = |\psi\rangle$

$\langle\beta_i|\psi\rangle = \sum_{i=1}^{n} |\beta_i\rangle\langle\beta_i|\psi\rangle \cdot$ (matrix; outer; vec)

$|\psi\rangle = \left(\sum_{i=1}^{n} |\beta_i\rangle\langle\beta_i|\right)|\psi\rangle \cdot$

$A|\psi\rangle = |\psi\rangle \quad \forall |\psi\rangle$

$\Rightarrow A = \mathbb{1}$

$\sum_{i=1}^{n} |\beta_i\rangle\langle\beta_i| = \mathbb{1}$

Vector representation in different bases

$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_n \end{pmatrix} = \sum_{i=1}^{n} \lambda_i |\beta_i\rangle = \sum_{i=1}^{n} |\beta_i\rangle\langle\beta_i|\psi\rangle$

$|\beta_i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} \langle\beta_1|\psi\rangle \\ \langle\beta_2|\psi\rangle \\ \langle\beta_n|\psi\rangle \end{pmatrix}$

Change of basis

$|\beta_i\rangle$

$|C_i\rangle$

$\sum_{i=1}^{n} |\beta_i\rangle\langle\beta_i| = \mathbb{1}$

$|\psi\rangle = \mathbb{1}|\psi\rangle$

$|\psi\rangle_B = \begin{pmatrix} \langle\beta_1|\psi\rangle \\ \vdots \\ \langle\beta_n|\psi\rangle \end{pmatrix}$ $\langle C_i|\psi\rangle = \langle C_i|\mathbb{1}|\psi\rangle = \sum_{j=1}^{n} \langle C_i|\beta_j\rangle\langle\beta_j|\psi\rangle$

$|\psi\rangle_C = \begin{pmatrix} \langle C_1|\psi\rangle \\ \vdots \\ \langle C_n|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle C_1|\beta_1\rangle & \cdots & \langle C_1|\beta_n\rangle \\ \vdots & & \vdots \\ \langle C_n|\beta_1\rangle & \cdots & \langle C_n|\beta_n\rangle \end{pmatrix} \begin{pmatrix} \langle\beta_1|\psi\rangle \\ \vdots \\ \langle\beta_n|\psi\rangle \end{pmatrix}$

$|\psi\rangle_C$ $P_{C \leftarrow B}$ $|\psi\rangle_B$

$|\psi\rangle_C = P_{C \leftarrow B}|\psi\rangle_B$

Multiplication and inverse of matrices

$AB = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$

$AB \neq BA$

$AA^{-1} = A^{-1}A = \mathbb{1}$

A is invertible
A^{-1} is the inverse of A.
otherwise A is singular.

$(AB)^{-1} = B^{-1}A^{-1}$