PHY 256 Lecture 3

Matrices inside inner products

$$\langle\psi|A|\phi\rangle = (\psi_1^* \ \psi_2^*)\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$$

$$= \psi_1^* A_{11}\phi_1 + \psi_2^* A_{21}\phi_1 + \psi_1^* A_{12}\phi_2 + \psi_2^* A_{22}\phi_2$$

$$(A|\phi\rangle)^\dagger = \langle\phi|A^\dagger$$

$$\langle\phi|A^\dagger|\psi\rangle = (\phi_1^* \ \phi_2^*)\begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$$

$$= \psi_1 A_{11}^*\phi_1^* + \psi_2 A_{21}^*\phi_1^* + \psi_1 A_{12}^*\phi_2^* + \psi_2 A_{22}^*\phi_2^*$$

$$\langle\psi|A|\phi\rangle^* = \langle\phi|A^\dagger|\psi\rangle$$

adjoint $\dagger$ rows $\leftrightarrow$ columns
complex conjugate

$|\psi\rangle^\dagger = \langle\psi|$ $\quad \langle\psi|^\dagger = |\psi\rangle$

$\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ $(\psi_1^* \ \psi_2^*)$ $\quad \begin{pmatrix} a & b \\ c & d\end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^*\end{pmatrix}$

$(a)^\dagger = (a^*)$

Eigenvalues and eigenvectors

$A|\psi\rangle = \lambda|\psi\rangle \quad \lambda\in\mathbb{C} \quad |\psi\rangle\neq 0$

$|\psi\rangle$ is an eigenvector of A
with the eigenvalue λ.

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} \quad |0\rangle = \begin{pmatrix} 1 \\ 0\end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1\end{pmatrix}$

$A|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix} = 1\cdot|0\rangle$ $\qquad \mathbb{C}^2 \quad \mathbb{C}^n$

$A|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix} = \begin{pmatrix} 0 \\ -1\end{pmatrix} = -1\cdot|1\rangle$

$|\psi\rangle \quad \alpha|\psi\rangle \ \forall\alpha \qquad |\psi\rangle = 3\cdot|0\rangle = \begin{pmatrix} 3 \\ 0\end{pmatrix}$

$A|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 3 \\ 0\end{pmatrix} = \begin{pmatrix} 3 \\ 0\end{pmatrix} = 1\cdot|\psi\rangle$

Hermitian Matrices

$A = A^\dagger$ self-adjoint

$\langle\psi|A|\phi\rangle^* = \langle\phi|A|\psi\rangle$

Real: $z^* = z$

$\langle\psi|A|\psi\rangle = \langle\psi|\lambda|\psi\rangle = \lambda\langle\psi|\psi\rangle = \lambda\,\underbrace{\|\psi\|^2}_{\mathbb{R}}$ $\quad \mathbb{R}$

$\langle\psi|A|\psi\rangle^* = \langle\psi|A|\psi\rangle \in\mathbb{R}$

$\lambda\in\mathbb{R}$

$\|\psi\|^2 = \langle\psi|\psi\rangle = (\psi_1^* \ \psi_2^*)\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \psi_1^*\psi_1 + \psi_2^*\psi_2 = |\psi_1|^2 + |\psi_2|^2$

$A|\psi\rangle = \lambda|\psi\rangle \qquad A|\phi\rangle = \mu|\phi\rangle$

$\lambda\neq\mu$

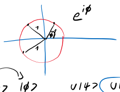
$\langle\psi|\phi\rangle = 0$

Hermitian matrix $\rightarrow$ orthonormal basis of eigenvectors
eigenbasis

Unitary Matrices

$U^\dagger = U^{-1} \qquad UU^\dagger = U^\dagger U = 1$

$|z|^2 = zz^* = 1$



$e^{i\phi}$

$U \quad |\psi\rangle \quad |\phi\rangle \qquad U|\psi\rangle \quad U|\phi\rangle$

$\langle\psi|\phi\rangle \qquad \langle\psi|U^\dagger$

$\langle\psi|\underbrace{U^\dagger U}_{1}|\phi\rangle = \langle\psi|\phi\rangle$

$e^{i\phi} \quad |\lambda| = 1$

Normal Matrices

$A^\dagger A = AA^\dagger$

Hermitian $A = A^\dagger$

Unitary $A^\dagger A = AA^\dagger = 1$

Matrix representation in a basis

$(A_{ij})_B = \langle B_i|A|B_j\rangle$

$A = \sum_{i=1}^n\sum_{j=1}^n (A_{ij})_B\,|B_i\rangle\langle B_j|$

Diagonalizable Matrices

A is normal

$\exists$ unitary P $\qquad P^\dagger AP = D$

D is diagonal: $\begin{pmatrix} D_1 & 0 \\ 0 & D_2\end{pmatrix}$

$A = \sum_{i=1}^n \lambda_i |B_i\rangle\langle B_i|$

The Cauchy-Schwarz inequality

$|\psi\rangle \quad |\phi\rangle$

$|\langle\psi|\phi\rangle| \le \|\psi\|\,\|\phi\|$

$|B_i\rangle \quad i = 1\dots n$

$|B_1\rangle \equiv \frac{|\phi\rangle}{\|\phi\|} \qquad \|B_1\| = 1$

Probability Theory

Random variables and probability distributions

$X\in\{1,\dots,6\}$

$X = 1$ heads
$\quad 0$ tails

$P(X = x) \qquad P(X=1) = \frac{1}{6}$

$[0,1]$

$P(X=0) = 0.45 \quad P(X=1) = 0.45$

0.9
90%

$P(X=x) = 0 \qquad P(X=x) = 1$

$P(X=1) = \frac{1}{6}$
$P(X=2) = \frac{1}{6}$
$\vdots$
$P(X=6) = \frac{1}{6}$ $\quad$ Uniform

$P(X=1) = 49.9\% = P(X=0)$

$P(X=\dots) = 0.2\%$

Loaded coin

$P(X=1) = \frac{1}{3} \qquad P(X=0) = \frac{2}{3}$ $\quad$ not uniform

fair

X_1	X_2	$X \equiv X_1 + X_2$	
0	0	0	$\frac{1}{4}$
0	1	1	$\frac{1}{2}$
1	0	1	
1	1	2	$\frac{1}{4}$

$\frac{1}{4}$

Conditional Probability

$X \quad Y$

$x_1,\dots,x_N \quad y_1,\dots,y_M$

Joint probability $x_i \ y_j$

$P(X = x_i \cap Y = y_j)$

$\sum_{j=1}^M P(X = x_i \cap Y = y_j) = P(X = x_i)$